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The q-Bessel Wavelet Packets

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Abstract Using the q-harmonic analysis associated with the q-Bessel operator, we study some types of q-wavelet packets and their corresponding q-wavelet transforms. We give for these wavelet transforms the related Plancheral and inversion formulas as well as their q-scale discrete scaling functions.

Keywords: q-Harmonic analysis, packets, wavelets.

1 Introduction

Contrary to what is prevalent and as far as we go back in the history of mathematics, we can say that the q-theory, also called Quantum Calculus, was initiated in 1748, when Euler undertook the study of the basic hypergeometric functions and considered the infinite product

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\dots}$$

as a generating function for $p(n)$, the number of partitions of a positive integer n into positive integers. But, it was not until a hundred years later that the subject acquired an independent status. This was first systematically effected by E. Heine (1821-1881) in the middle of the nineteenth century, and the work was subsequently greatly extended by F. H. Jackson (1870-1960), W. N. Bailey (1893-1961), L. J. Slater, G. E. Andrews and many others up to the present day. In fact, in recent years, various families of q-series and q-polynomials have been investigated rather widely and extensively due mainly to their having been found to be potentially useful in such wide variety of fields as (for example) theory of partitions, number theory, combinatorial analysis, finite vector spaces, Lie theory, particle physics, nonlinear electric circuit theory, mechanical engineering, theory of heat conduction, quantum mechanics, cosmology and statistics. The books and monographs by (among others) H. M. Srivastava and J. Choi [25], H. Exton [8], H. M. Srivastava and P. W. Karlsson [26] and G. Gasper and M. Rahman [14] discussed extensively basic (or q-) hypergeometric functions in one, two and more variables. The success of the theory began since the 1970's, and the q-theory became an active area of research. This success was achieved thanks to the work of G. E. Andrews and R. Askey ([1], [2]) on the orthogonal polynomials, special functions and their q-analogues. Since then, many authors have been interested by the theory and many papers have been published (see [9], [21], [24]). In particular, in [20], T. H. Koornwinder and R. Swarttouw studied the third Jackson q-Bessel function and claimed that we can use it to build a reliable harmonic analysis. This motivated and encouraged many authors to study elements of q-harmonic analysis associated to different q-difference-differential operators and publish many papers in the subject (see [3], [4], [5], [13], [22], [23] and references therein).
Since the classical harmonic analysis plays central role in the theory of wavelets and wavelet packets, it is natural to ask if we can apply the q-harmonic analysis to build new wavelets and wavelets packets. Many papers treated the notion of q-wavelets (see [10], [11], [12] and references therein) and gave some applications using q-harmonic analysis.
In this paper, we are concerned with the notion of q-wavelet packets. We shall use the harmonic analysis associated with the q-Bessel operator presented in [4] and [13] to study some types of q-wavelet packets following the ideas presented in [27].

This paper is organized as follows : in Section 2, we present some notations and notions from the quantum calculus needed in the sequel. Section 3 is devoted to recalling some elements of harmonic analysis associated with the q-Bessel operator. In Section 4, we introduce and study the q-Bessel wavelet packets and its related transform. Finally, in Section 5, we introduce and study the Bessel's q-scale discrete scaling function and its related transform.

2 Notations and Preliminaries

We recall some usual notions and notations used in the q-theory (see [14], [19] and [[25], Chapter 6]). We refer to the book by G. Gasper and M. Rahman [14] and [[25], Chapter 6]) for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions.
Throughout this paper, we assume $q \in [0,1]$ and we denote

$$\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}, \quad \mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\} \quad \text{and} \quad \widetilde{\mathbb{R}}_{q,+} = \mathbb{R}_{q,+} \cup \{0\}.$$

For a complex number a, the q-shifted factorials are defined by:

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1,2,...; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We also write

$$[x]_q = \frac{1-q^x}{1-q}, \qquad x \in \mathbb{C} \quad \text{and} \quad n!_q = \frac{(q;q)_n}{(1-q)^n}, \qquad n \in \mathbb{N}.$$

The Rubin's q-differential operator is defined in [22] and [23] by

$$\partial_q(f)(z) = \begin{cases} \dfrac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & if\ z \neq 0 \\ \lim\limits_{x\to 0} \partial_q(f)(x) & if\ z = 0. \end{cases} \tag{2.1}$$

Note that if f is differentiable at z, then $\partial_q(f)(z)$ tend to $f'(z)$ as q tends to 1.

The Jackson's q-integrals from 0 to a and from 0 to $+\infty$ (see [18]) are given by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n \quad \text{and} \quad \int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n \tag{2.2}$$

provided the sums converge absolutely.
The Jackson's q-integral in a generic interval $[a,b]$ is given by

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.3}$$

Remark that in the particular case $a = bq^n$, $n \in \mathbb{N}$, the relation (2.3) becomes

$$\int_a^b f(x)d_qx = (1-q)b\sum_{k=0}^{n-1} f(q^kb)q^k. \tag{2.4}$$

In the sequel, we will need the following sets and spaces.
• $\mathcal{C}_{q,0}(\mathbb{R}_q)$ the space of bounded functions on $\mathbb{R}_q$, which are continuous at 0 and vanishing at ∞.
• $\mathcal{S}_{*,q}(\mathbb{R}_q)$ the space of even functions f defined on $\mathbb{R}_q$ satisfying

$$\forall n,m \in \mathbb{N}, \quad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q} \mid x^m \partial_q^n f(x) \mid < +\infty$$

and

$$\lim_{x\to 0} \partial_q^n f(x) \quad (\text{in} \quad \mathbb{R}_q) \qquad \text{exists.}$$

• $\mathcal{D}_{*,q}(\mathbb{R}_q)$ the subspace of $\mathcal{S}_*, q(\mathbb{R}_q)$ composed of functions with compact supports.
• $L^p_{\alpha,q}(\mathbb{R}_{q,+}) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_0^\infty |f(x)|^p |x|^{2\alpha+1} d_qx \right)^{\frac{1}{p}} < \infty \right\}, \quad p > 0$ and $\alpha \in \mathbb{R}$.
• $L^\infty_q(\mathbb{R}_{q,+}) = \left\{ f : \|f\|_{\infty,q} = \sup\limits_{x\in\mathbb{R}_{q,+}} |f(x)| < \infty \right\}.$

3 *q*-Bessel Fourier Transform

The normalized third Jackson's q-Bessel function is defined by

$$j_\alpha(x;q^2)=\sum_{n=0}^{+\infty}(-1)^n\frac{\Gamma_{q^2}(\alpha+1)q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)}\left(\frac{x}{1+q}\right)^{2n}, \tag{3.1}$$

where

$$\Gamma(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\quad x\neq 0,-2,-4,-6,\dots$$

is the q-Gamma function. Note that we have

$$j_\alpha(x;q^2)=(1-q^2)^\alpha\Gamma_{q^2}(\alpha+1)\left((1-q)x\right)^{-\alpha}J_\alpha((1-q)x;q^2), \tag{3.2}$$

where

$$J_\alpha(x;q^2)=\frac{x^\alpha(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}\cdot{}_1\varphi_1(0;q^{2\alpha+2};q^2,q^2x^2) \tag{3.3}$$

is the third Jackson's q-Bessel function.
For $\alpha\geq-\dfrac{1}{2}$,

$$j_\alpha(.;q^2)\in\mathcal{S}_{*,q}(\mathbb{R}_q)$$

and using the relations (3.1), we obtain

$$\partial_q j_\alpha(x;q^2)=-\frac{x}{[2\alpha+2]_q}j_{\alpha+1}(x;q^2). \tag{3.4}$$

As a consequence, we have

Proposition 1 *For $\lambda\in\mathbb{C}$, the function $j_\alpha(\lambda x;q^2)$ is the unique even solution of the problem*

$$\begin{cases}\triangle_{\alpha,q}f(x)=-\lambda^2f(x),\\ f(0)=1,\end{cases} \tag{3.5}$$

where $\triangle_{\alpha,q}$ is the q-Bessel operator, given by

$$\triangle_{\alpha,q}f(x)=\frac{1}{|x|^{2\alpha+1}}\partial_q[|x|^{2\alpha+1}\partial_qf(x)].$$

Proposition 2 *For $x,y\in\mathbb{R}_{q,+}$, we have*

$$(xy)^{\alpha+1}\int_0^{+\infty}j_\alpha(xt;q^2)j_\alpha(yt;q^2)t^{2\alpha+1}d_qt=\frac{(1+q)^{2\alpha}\Gamma_{q^2}^2(\alpha+1)}{(1-q)}\delta_{x,y}. \tag{3.6}$$

Definition 1 *The q-Bessel Fourier transform is defined for $f\in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, by*

$$\mathcal{F}_{\alpha,q}(f)(\lambda)=c_{\alpha,q}\int_0^\infty f(x)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx, \tag{3.7}$$

where

$$c_{\alpha,q}=\frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}. \tag{3.8}$$

Letting $q\uparrow 1$ be subject to the condition $\dfrac{Log(1-q)}{Log(q)}\in 2\mathbb{Z}$, gives, at least formally, the classical Bessel-Fourier transform.
Some properties of the q-Bessel Fourier transform are given in the following result (see [4]).

Theorem 1 *1) For $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, we have $\mathcal{F}_{\alpha,q}(f) \in L^\infty_q(\mathbb{R}_{q,+})$,*

$$\lim_{\lambda\to+\infty} \mathcal{F}_{\alpha,q}(f)(\lambda) = 0 \qquad \text{and} \qquad \|\mathcal{F}_{\alpha,q}(f)\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q;q)_\infty}\|f\|_{1,\alpha,q}.$$

2) For $f, g \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, we have

$$\int_0^\infty f(x)\mathcal{F}_{\alpha,q}(g)(x)x^{2\alpha+1}d_qx = \int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda)g(\lambda)\lambda^{2\alpha+1}d_q\lambda. \tag{3.9}$$

3) If f and $\triangle_{\alpha,q}f$ are in $L^1_{\alpha,q}(\mathbb{R}_{q,+})$, then

$$\mathcal{F}_{\alpha,q}(\triangle_{\alpha,q}f)(\lambda) = -\lambda^2\mathcal{F}_{\alpha,q}(f)(\lambda).$$

4) If f and x^2f are in $L^1_{\alpha,q}(\mathbb{R}_{q,+})$, then

$$\triangle_{\alpha,q}(\mathcal{F}_{\alpha,q}(f)) = -\mathcal{F}_{\alpha,q}(x^2f).$$

Proposition 3 *If $f \in L^1_{\alpha,q}(\mathbb{R}_{q,+})$, then*

$$\forall x \in \mathbb{R}_{q,+}, \qquad f(x) = c_{\alpha,q}\int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda)j_\alpha(\lambda x;q^2)\lambda^{2\alpha+1}d_q\lambda.$$

Theorem 2 *1) Plancherel formula*
For all $f \in \mathcal{S}_{*,q}(\mathbb{R}_q)$, we have $\mathcal{F}_{\alpha,q}(f) \in \mathcal{S}_{*,q}(\mathbb{R}_q)$ and

$$\|\mathcal{F}_{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}. \tag{3.10}$$

2) Plancherel theorem
The q-Bessel transform can be uniquely extended to an isometric isomorphism on $L^2_{\alpha,q}(\mathbb{R}_{q,+})$ with $\mathcal{F}^{-1}_{\alpha,q} = \mathcal{F}_{\alpha,q}$.

We are now in a position to define the generalized q-Bessel translation operator.

Definition 2 *The generalized q-Bessel translation operator is defined for $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$ by*

$$T^{\alpha;q}_y(f)(x) = c_{\alpha,q}\int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda)j_\alpha(\lambda x;q^2)j_\alpha(\lambda y;q^2)\lambda^{2\alpha+1}d_q\lambda, \quad x,y \in \mathbb{R}_{q,+}, \tag{3.11}$$

$$T^{\alpha;q}_0(f) = f.$$

It verifies the following properties.

Proposition 4

1. *For all $x, y \in \mathbb{R}_{q,+}$, $T^{\alpha;q}_y(f)(x) = T^{\alpha;q}_x(f)(y)$.*

2. *For $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+}), x, y \in \mathbb{R}_{q,+}$, we have*

$$\mathcal{F}_{\alpha,q}(T^{\alpha;q}_y f)(\lambda) = j_\alpha(\lambda y;q^2)\mathcal{F}_{\alpha,q}(f)(\lambda). \tag{3.12}$$

3. *If $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$ (resp. $\mathcal{S}_{*,q}(\mathbb{R}_q)$) then $T^{\alpha;q}_y(f) \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$ (resp. $\mathcal{S}_{*,q}(\mathbb{R}_q)$) and we have*

$$\| T^{\alpha;q}_y(f) \|_{2,\alpha,q} \leq \frac{4}{(q;q)_\infty} \| f \|_{2,\alpha,q} . \tag{3.13}$$

4. *For all $x, y, \lambda \in \mathbb{R}_{q,+}$, $T^{\alpha;q}_y\left(j_\alpha(\lambda\cdot;q^2)\right)(x) = j_\alpha(\lambda x;q^2)j_\alpha(\lambda y;q^2)$.*

Definition 3 *The q-Bessel convolution product is defined for* $f, g \in \mathcal{S}_{*,q}(\mathbb{R}_q)$ *by:*

$$f *_B g(x) = c_{\alpha,q} \int_0^\infty T_x^{\alpha;q} f(y) g(y) y^{2\alpha+1} d_q y. \tag{3.14}$$

In the following propositions, we present some of its properties.

Proposition 5 *For* $f, g \in \mathcal{S}_{*,q}(\mathbb{R}_q)$*, we have*

1. $\mathcal{F}_{\alpha,q}(f *_B g) = \mathcal{F}_{\alpha,q}(f).\mathcal{F}_{\alpha,q}(g)$.
2. $f *_B g = g *_B f$.
3. $(f *_B g) *_B h = f *_B (g *_B h)$.

Proposition 6 *Let* f *and* g *be in* $\mathcal{S}_{*,q}(\mathbb{R}_q)$*. Then* $f *_B g \in \mathcal{S}_q(\mathbb{R}_q)$*, and*

$$\|f *_B g\|_{2,\alpha,q} = \|\mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(g)\|_{2,\alpha,q} \tag{3.15}$$

We finish this section by the following useful result.

Proposition 7 *For* $a \in \mathbb{R}_{q,+}$ *the operator* H_a *defined for* $g \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$ *(resp.* $\mathcal{S}_{*,q}(\mathbb{R}_q)$*), by*

$$H_a(g)(x) = \frac{1}{a^{2\alpha+2}} g\left(\frac{x}{a}\right)$$

is linear and bijective from $L^2_{\alpha,q}(\mathbb{R}_{q,+})$ *(resp.* $\mathcal{S}_{*,q}(\mathbb{R}_q)$*) into itself and we have*

$$\|H_a(g)\|_{2,\alpha,q} = \frac{1}{a^{\alpha+1}} \|g\|_{2,\alpha,q} \tag{3.16}$$

and

$$\mathcal{F}_{\alpha,q}(H_a(g))(\lambda) = \mathcal{F}_{\alpha,q}(g)(a\lambda), \quad \lambda \in \widetilde{\mathbb{R}}_q. \tag{3.17}$$

Proof
The linearity and the bijectivity of H_a are clear. In Particular, $H_a^{-1} = H_{\frac{1}{a}}$. The change of variables $u = \dfrac{x}{a}$ completes the proof of the result. □

4 q-Bessel Wavelet Packets

We recall that a Bessel's q-wavelet is a square q-integrable function g on $\mathbb{R}_{q,+}$ satisfying the following admissibility condition (see [11]):

$$0 < C_g = \int_0^\infty \mid \mathcal{F}_{\alpha,q}(g)(a) \mid^2 \frac{d_q a}{a} < \infty. \tag{4.1}$$

We consider a Bessel's q-wavelet g and a strictly decreasing scale sequence $(r_j)_{j\in\mathbb{Z}}$ of $\mathbb{R}_{q,+}$ satisfying $\lim_{j\to-\infty} r_j = +\infty$, $\lim_{j\to+\infty} r_j = 0$. We state the following introductory result.

Proposition 8 *For all* $j \in \mathbb{Z}$*, we have :*

1. *the function* $\lambda \longmapsto \left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j} |\mathcal{F}_{\alpha,q}(H_a(g))(\lambda)|^2 \frac{d_q a}{a}\right)^{\frac{1}{2}}$ *belongs to* $L^2_{\alpha,q}(\mathbb{R}_{q,+})$*,*

2. there exists a function $g_j^P \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$ *such that for all* $\lambda \in \mathbb{R}_{q,+}$,

$$\mathcal{F}_{\alpha,q}(g_j^P)(\lambda) = \left(\frac{1}{C_g} \int_{r_{j+1}}^{r_j} |\mathcal{F}_{\alpha,q}(H_a(g))(\lambda)|^2 \frac{d_q a}{a} \right)^{\frac{1}{2}}.$$

Proof

Fix $j \in \mathbb{Z}$.

(1) On the one hand, r_j and r_{j+1} are two elements of $\mathbb{R}_{q,+}$ satisfying $r_{j+1} < r_j$, then there exists a positive integer n such that $r_{j+1} = q^n r_j$. So, using the relation (2.4) and Proposition 7, we obtain

$$\begin{aligned} \int_0^\infty \left(\frac{1}{C_g} \int_{r_{j+1}}^{r_j} |\mathcal{F}_{\alpha,q}(H_a(g))(\lambda)|^2 \frac{d_q a}{a} \right) \lambda^{2\alpha+1} d_q\lambda &= \frac{1-q}{C_g} \int_0^\infty \sum_{k=0}^{n-1} |\mathcal{F}_{\alpha,q}(g)(\lambda q^k r_j)|^2 \lambda^{2\alpha+1} d_q\lambda \\ &= \frac{1-q}{C_g} \sum_{k=0}^{n-1} \int_0^\infty |\mathcal{F}_{\alpha,q}(g)(\lambda q^k r_j)|^2 \lambda^{2\alpha+1} d_q\lambda. \end{aligned}$$

On the other hand, the change of variable $u = \lambda q^k r_j$, ($0 \le k \le n-1$), together with Theorem 2 leads to

$$\begin{aligned} \int_0^\infty \left(\frac{1}{C_g} \int_{r_{j+1}}^{r_j} |\mathcal{F}_{\alpha,q}(H_a(g))(\lambda)|^2 \frac{d_q a}{a} \right) \lambda^{2\alpha+1} d_q\lambda &= \frac{1-q}{C_g} \sum_{k=0}^{n-1} \int_0^\infty \frac{|\mathcal{F}_{\alpha,q}(g)(u)|^2}{(r_j q^k)^{2\alpha+2}} u^{2\alpha+1} d_q u \\ &= \frac{1-q}{C_g} \|\mathcal{F}_{\alpha,q}(g)(u)\|^2_{2,\alpha,q} \sum_{k=0}^{n-1} \frac{1}{(r_j q^k)^{2\alpha+2}} \\ &= \frac{q^{2\alpha+2}}{C_g [2\alpha+2]_q} \left(\frac{1}{r_{j+1}^{2\alpha+2}} - \frac{1}{r_j^{2\alpha+2}} \right) \|g\|^2_{2,\alpha,q}. \end{aligned}$$

(2) The result follows from Theorem 2. □

Definition 4 *i) The sequence* $(g_j^P)_{j\in\mathbb{Z}}$ *is called Bessel's q-wavelet packet.*
ii) The function g_j^P, $j \in \mathbb{Z}$, *is called Bessel's q-wavelet packet's member of step j.*

We have the following immediate properties.

Proposition 9 *For all* $\lambda \in \mathbb{R}_q$, *we have*

$$0 \le \mathcal{F}_{\alpha,q}(g_j^P)(\lambda) \le 1, \quad j \in \mathbb{Z} \qquad \text{and} \qquad \sum_{j=-\infty}^{+\infty} [\mathcal{F}_{\alpha,q}(g_j^P)(\lambda)]^2 = 1.$$

Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Bessel's q-wavelet packet. We introduce for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}_{q,+}$, the function $g_{j,x}^P$ as

$$g_{j,x}^P(y) = T_y^{\alpha;q}(g_j^P)(x), \quad y \in \widetilde{\mathbb{R}}_{q,+}. \tag{4.2}$$

Some properties of these functions are summarized in the following result and its proof follows easily from the properties of the q-Bessel translation operator and the definition of the Bessel's q-wavelet packets.

Proposition 10 *For all* $j \in \mathbb{Z}$ *and* $x \in \mathbb{R}_{q,+}$, *the function* $g_{j,x}^P$ *belongs to* $L^2_{\alpha,q}(\mathbb{R}_{q,+})$ *and we have for all* $\lambda \in \widetilde{\mathbb{R}}_{q,+}$,

- $\mathcal{F}_{\alpha,q}(g_{j,x}^P)(\lambda) = j_\alpha(\lambda x; q^2)\mathcal{F}_{\alpha,q}(g_j^P)(\lambda)$.
- $\|g_{j,x}^P\|_{2,\alpha,q} \le \dfrac{4\|g_j^P\|_{2,\alpha,q}}{(q;q)_\infty}$.

Definition 5 *Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Bessel's q-wavelet packet. We define the Bessel's q-wavelet packet transform $\Psi_{q,g}^P$ by*

$$\Psi_{q,g}^P(f)(j,y) = c_{\alpha,q}\int_0^\infty f(x)\overline{g_{j,y}^P}(x)x^{2\alpha+1}d_qx,\ j\in\mathbb{Z},\quad y\in\widetilde{\mathbb{R}}_{q,+}\quad and\quad f\in L^2_{\alpha,q}(\mathbb{R}_{q,+}), \tag{4.3}$$

where $c_{\alpha,q}$ is given by the relation (3.8).

Remark 1 *The equality (4.3) is equivalent to*

$$\Psi_{q,g}^P(f)(j,y) = f *_B \overline{g_j^P}(y) = \mathcal{F}_{\alpha,q}(\mathcal{F}_{\alpha,q}(f *_B \overline{g_j^P}))(y) = \mathcal{F}_{\alpha,q}[\mathcal{F}_{\alpha,q}(f).\mathcal{F}_{\alpha,q}(\overline{g_j^P})](y). \tag{4.4}$$

The following proposition provides some useful properties of $\Psi_{q,g}^P$.

Proposition 11 *Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Bessel's q-wavelet packet and $f\in L^2_{\alpha,q}(\mathbb{R}_{q,+})$. Then,*

1. *for all $j\in\mathbb{Z}$, $b\in\widetilde{\mathbb{R}}_{q,+}$, we have*

$$|\Psi_{q,g}^P(f)(j,b)| \le \frac{4c_{\alpha,q}}{(q;q)_\infty}\|f\|_{2,\alpha,q}\left\|g_j^P\right\|_{2,\alpha,q};$$

2. *for all $j\in\mathbb{Z}$, the mapping $b\mapsto\Psi_{q,g}^P(f)(j,b)$ is continuous on $\widetilde{\mathbb{R}}_{q,+}$ and we have $\lim_{b\to\infty}\Psi_{q,g}^P(f)(j,b)=0$.*

Proof
(1) From the relation (4.3), Proposition 10 and the Cauchy-Schwarz inequality, we have for $j\in\mathbb{Z}$ and $b\in\mathbb{R}_{q,+}$

$$|\Psi_{q,g}^P(f)(j,b)| = c_{\alpha,q}\left|\int_0^\infty f(x)\overline{g_{j,b}^P}(x)x^{2\alpha+1}d_qx\right| \le c_{\alpha,q}||f||_{2,\alpha,q}||g_{j,b}^P||_{2,\alpha,q} \le \frac{4c_{\alpha,q}}{(q;q)_\infty}||f||_{2,\alpha,q}||g_j^P||_{2,\alpha,q}.$$

(2) Let $j\in\mathbb{Z}$ and $f\in L^2_{\alpha,q}(\mathbb{R}_{q,+})$. From Theorem 2, we have $\mathcal{F}_{\alpha,q}(f)$ and $\mathcal{F}_{\alpha,q}(\overline{g_j^P})$ are in $L^2_{\alpha,q}(\mathbb{R}_{q,+})$ and the product $\mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(\overline{g_j^P})$ is in $L_{1,\alpha,q}(\mathbb{R}_{q,+})$. So, the relation (4.4) together with Theorem 1 achieves the proof. □

The following result shows Plancheral and Parseval formulas for the Bessel's q-wavelet packet transform $\Psi_{q,g}^P$.

Theorem 3 *Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Bessel's q-wavelet packet.*
(1) **Plancheral formula for $\Psi_{q,g}^P$**
For $f\in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, we have

$$\sum_{j=-\infty}^{+\infty}\int_0^\infty |\ \Psi_{q,g}^P(f)(j,b)\ |^2\ b^{2\alpha+1}d_qb = \|f\|^2_{2,\alpha,q}. \tag{4.5}$$

(2) **Parseval formula for $\Psi_{q,g}^P$**
For $f_1, f_2\in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, we have

$$\int_0^\infty f_1(x)\overline{f_2(x)}x^{2\alpha+1}d_qx = \sum_{j=-\infty}^{+\infty}\int_0^\infty \Psi_{q,g}^P(f_1)(j,b)\overline{\Psi_{q,g}^P(f_2)(j,b)}b^{2\alpha+1}d_qb. \tag{4.6}$$

Proof
(1) From the relations (3.15) and (4.4), we obtain

$$\int_0^\infty |\Psi_{q,g}^P(f)(j,b)|^2 b^{2\alpha+1}d_qb = \int_0^\infty |\mathcal{F}_{\alpha,q}(f)(a)|^2[\mathcal{F}_{\alpha,q}(g_j^P(a)]^2a^{2\alpha+1}d_qa.$$

So, the use of the Fubini's theorem and the fact that

$$\sum_{j=-\infty}^{+\infty} [\mathcal{F}_{\alpha,q}(g_j^P)(\lambda)]^2 = 1$$

give

$$\begin{aligned}\sum_{j=-\infty}^{+\infty} \int_0^{\infty} |\Psi_{q,g}^P(f)(j,b)|^2 b^{2\alpha+1} d_q b &= \int_0^{\infty} |\mathcal{F}_{\alpha,q}(f)(a)|^2 \sum_{j=-\infty}^{+\infty} [\mathcal{F}_{\alpha,q}(g_j^P(a))]^2 a^{2\alpha+1} d_q a \\ &= \int_0^{\infty} |\mathcal{F}_{\alpha,q}(f)(a)|^2 a^{2\alpha+1} d_q a.\end{aligned}$$

Thus, (4.5) follows from Theorem 2.
(2) The result is a direct consequence of assertion (1). □

Theorem 4 *Let $(g_j^P)_{j\in\mathbb{Z}}$ be a Bessel's q-wavelet packet. For $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, one has the following reconstruction formula :*

$$f(x) = c_{\alpha,q} \sum_{j=-\infty}^{+\infty} \int_0^{\infty} \Psi_{q,g}^P(f)(j,b) g_{j,b}^P(x) b^{2\alpha+1} d_q b, \quad x \in \mathbb{R}_{q,+}.$$

Proof
For $x \in \mathbb{R}_{q,+}$, we have $h = \delta_x$ belongs to $L^2_{\alpha,q}(\mathbb{R}_{q,+})$. Then, according to the relation (4.6), the definition of $\Psi_{q,g}^P$ and the definition of the Jackson's q-integral, we have

$$\begin{aligned}(1-q)x^{2\alpha+2} f(x) &= \sum_{j=-\infty}^{\infty} \int_0^{\infty} \Psi_{q,g}^P(f)(j,b)\overline{\Psi_{q,g}^P(h)}(j,b) b^{2\alpha+1} d_q b \\ &= c_{\alpha,q} \sum_{j=-\infty}^{\infty} \int_0^{\infty} \Psi_{q,g}^P(f)(j,b) \left(\int_0^{\infty} \overline{h}(t) g_{j,b}^P(t) |t|^{2\alpha+1} d_q t\right) b^{2\alpha+1} d_q b \\ &= (1-q)x^{2\alpha+2} c_{\alpha,q} \sum_{j=-\infty}^{\infty} \int_0^{\infty} \Psi_{q,g}^P(f)(j,b) g_{j,b}^P(x) b^{2\alpha+1} d_q b,\end{aligned}$$

which is equivalent to

$$f(x) = c_{\alpha,q} \sum_{j=-\infty}^{\infty} \int_0^{\infty} \Psi_{q,g}^P(f)(j,b) g_{j,b}^P(x) b^{2\alpha+1} d_q b.$$

□

5 Bessel's *q*-Scale Discrete Scaling Function

In this section, we consider a Bessel's q-wavelet packet $(g_j^P)_{j\in\mathbb{Z}}$.

Proposition 12

1. *For all $m \in \mathbb{Z}$ and $x \in \mathbb{R}_{q,+}$, we have*

$$\sum_{j=-\infty}^{m-1} [\mathcal{F}_{\alpha,q}(g_j^P)(x)]^2 = \frac{1}{C_g} \int_{r_m}^{\infty} |\mathcal{F}_{\alpha,q}(H_a(g))(x)|^2 \frac{d_q a}{a}. \quad (5.1)$$

2. For all $m \in \mathbb{Z}$, the function $x \mapsto \left(\sum_{j=-\infty}^{m-1} [\mathcal{F}_{\alpha,q}(g_j^P)(x)]^2 \right)^{\frac{1}{2}}$ *belongs to* $L^2_{\alpha,q}(\mathbb{R}_{q,+})$.

3. For all $m \in \mathbb{Z}$ there exists a function G_m^P in $L^2_{\alpha,q}(\mathbb{R}_{q,+})$ such that for all $x \in \mathbb{R}_{q,+}$,

$$\mathcal{F}_{\alpha,q}(G_m^P)(x) = \left(\sum_{j=-\infty}^{m-1} [\mathcal{F}_{\alpha,q}(g_j^P)(x)]^2 \right)^{\frac{1}{2}}. \tag{5.2}$$

Proof
(1) It follows from the definition of g_j^P.
(2) From the Fubini's theorem, the relation (5.1) and Proposition 7, we have

$$\begin{aligned}\int_0^\infty \sum_{j=-\infty}^{m-1} [\mathcal{F}_{\alpha,q}(g_j^P)(x)]^2 x^{2\alpha+1} d_qx &= \frac{1}{C_g}\int_0^\infty \int_{r_m}^\infty |\mathcal{F}_{\alpha,q}(H_a(g))(x)|^2 \frac{d_qa}{a} x^{2\alpha+1} d_qx \\ &= \frac{1}{C_g}\int_{r_m}^\infty \left(\int_0^\infty |\mathcal{F}_{\alpha,q}(g)(ax)|^2 x^{2\alpha+1} d_qx \right) \frac{d_qa}{a}.\end{aligned}$$

By the change of variables $u = ax$ and Theorem 2, we obtain

$$\begin{aligned}\int_0^\infty \left(\frac{1}{C_g}\int_{r_m}^\infty |\mathcal{F}_{\alpha,q}(H_a(g))(x)|^2 \frac{d_qa}{a} \right) x^{2\alpha+1} d_qx &= \frac{1}{C_g}\int_{r_m}^\infty \left(\int_0^\infty |\mathcal{F}_{\alpha,q}(g)(x)|^2 x^{2\alpha+1} d_qx \right) \frac{d_qa}{a^{2\alpha+3}} \\ &= \frac{\|g\|_{2,\alpha,q}}{C_g} \int_{r_m}^\infty \frac{d_qa}{a^{2\alpha+3}} < \infty.\end{aligned}$$

This completes the proof of (2).
(3) We deduce the result from the previous assertion and Theorem 2. □

Definition 6 *The sequence $(G_m^P)_{m\in\mathbb{Z}}$ is called Bessel's q-scale discrete scaling function.*

The sequence $(G_m^P)_{m\in\mathbb{Z}}$ verifies the following trivial and easily proved properties.

Proposition 13
(i) For all $m \in \mathbb{Z}$ and $\lambda \in \mathbb{R}_{q,+}$, we have

$$0 \leq \mathcal{F}_{\alpha,q}(G_m^P)(\lambda) \leq 1. \tag{5.3}$$

(ii) For all $\lambda \in \mathbb{R}_{q,+}$, we have

$$\lim_{m\to+\infty} \mathcal{F}_{\alpha,q}(G_m^P)(\lambda) = 1. \tag{5.4}$$

Proof
The proof is an easy deduction from Proposition 9. □

Proposition 14 *For $m \in \mathbb{Z}$ and $x \in \mathbb{R}_{q,+}$, the following relations*
(i)

$$\left[\mathcal{F}_{\alpha,q}(G_m^P)(x)\right]^2 + \sum_{j=m}^{\infty} [\mathcal{F}_{\alpha,q}(g_j^P)(x)]^2 = 1, \tag{5.5}$$

(ii)

$$\left[\mathcal{F}_{\alpha,q}(g_m^P)(x)\right]^2 = \left[\mathcal{F}_{\alpha,q}(G_{m+1}^P)(x)\right]^2 - \left[\mathcal{F}_{\alpha,q}(G_m^P)(x)\right]^2, \tag{5.6}$$

(iii)

$$\sum_{m=-\infty}^{\infty}\left(\left[\mathcal{F}_{\alpha,q}(G^P_{m+1})(x)\right]^2-\left[\mathcal{F}_{\alpha,q}(G^P_m)(x)\right]^2\right)=1 \tag{5.7}$$

hold.

Proof

- (i) It follows immediately from (5.2) and Proposition 9.
- (ii) We deduce the result from the relation (5.2).
- (iii) The relation is a consequence of (5.6) and Proposition 9.

□

Now, let $(G^P_m)_{m\in\mathbb{Z}}$ be a Bessel's q-scale discrete scaling function and consider for all $m\in\mathbb{Z},\ \ x\in\mathbb{R}_{q,+}$, the function $G^P_{m,x}$ is given by

$$G^P_{m,x}(y)=T^{\alpha,q}_y(G^P_m)(x),\quad \forall y\in\mathbb{R}_{q,+}. \tag{5.8}$$

From the properties of the Bessel's q-translation, one can prove easily the following result giving some properties of the function $G^P_{m,x}$.

Proposition 15 *For all $m\in\mathbb{Z}$ and $x\in\mathbb{R}_{q,+}$, the function $G^P_{m,x}$ belongs to $L^2_{\alpha,q}(\mathbb{R}_{q,+})$ and we have*

- $\mathcal{F}_{\alpha,q}(G^P_{m,x})(\lambda)=j_\alpha(\lambda x;q^2)\mathcal{F}_{\alpha,q}(G^P_m)(\lambda),\quad \lambda\in\mathbb{R}_{q,+},$
- $\|G^P_{m,x}\|_{2,\alpha,q}\le\dfrac{4\|G^P_m\|_{2,\alpha,q}}{(q;q)_\infty}.$

Definition 7 *Let $(G^P_m)_{m\in\mathbb{Z}}$ be a Bessel's q-scale discrete scaling function. We define the Bessel's q-scale discrete scaling transform $\Theta^P_{q,G}$ on $L^2_{\alpha,q}(\mathbb{R}_{q,+})$, by*

$$\Theta^P_{q,G}(f)(m,x)=c_{\alpha,q}\int_0^\infty f(b)\overline{G^P_{m,x}(b)}b^{2\alpha+1}d_qb,\quad m\in\mathbb{Z},\ \text{and}\ \ x\in\mathbb{R}_{q,+}. \tag{5.9}$$

Remark 2 *The relation (5.9) is equivalent to*

$$\Theta^P_{q,G}(f)(m,x)=f*_B\overline{G^P_m}(x). \tag{5.10}$$

In the two following results, we will provide a Plancheral and a Parseval formulas for $\Theta^P_{q,G}$.

Theorem 5 *Let $(G^P_m)_{m\in\mathbb{Z}}$ be a Bessel's q-scale discrete scaling function.*
(1) **Plancherel formula for $\Theta^P_{q,G}$**
For $f\in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, we have

$$\|f\|^2_{2,\alpha,q}=\lim_{m\to+\infty}\int_0^\infty|\Theta^P_{q,G}(f)(m,b)|^2b^{2\alpha+1}d_qb. \tag{5.11}$$

(2) **Parseval formula for $\Theta^P_{q,G}$**
For $f_1,f_2\in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, we have

$$\int_0^\infty f_1(x)\overline{f}_2(x)x^{2\alpha+1}d_qx=\lim_{m\to+\infty}\int_0^\infty\Theta^P_{q,G}(f_1)(m,b)\overline{\Theta^P_{q,G}(f_2)(m,b)}b^{2\alpha+1}d_qb. \tag{5.12}$$

Proof
(1) Due to the relations (5.10) and (3.15), we have for all $m \in \mathbb{Z}$,

$$\int_0^\infty |\Theta_{q,G}^P(f)(m,b)|^2 b^{2\alpha+1} d_q b = \int_0^\infty |\mathcal{F}_{\alpha,q}(f)(x)|^2 [\mathcal{F}_{\alpha,q}(G_m^P)(x)]^2 x^{2\alpha+1} d_q x. \tag{5.13}$$

The relations (5.3) and (5.4), and the Lebesgue's theorem yield to

$$\lim_{m\to+\infty} \int_0^\infty |\Theta_{q,G}^P(f)(m,b)|^2 b^{2\alpha+1} d_q b = \|\mathcal{F}_{\alpha,q}(f)\|_{2,\alpha,q}^2 .$$

Finally, Theorem 2 achieves the proof of (1).
(2) The result follows from (5.11). □

Using the Bessel's q-scale discrete scaling function $(G_m^P)_{m\in\mathbb{Z}}$ and the Bessel's q-wavelet packet transform $\Psi_{q,g}^P$, one can obtain another Plancheral formula for $\Theta_{q,G}^P$. This is the aim of the following result.

Theorem 6
(1) **Plancherel formula for $\Theta_{q,G}^P$ using $\Psi_{q,g}^P$**
For all $f \in L_{\alpha,q}^2(\mathbb{R}_{q,+})$, we have for all $m \in \mathbb{Z}$,

$$\|f\|_{2,\alpha,q}^2 = \int_0^\infty |\Theta_{q,G}^P(f)(m,b)|^2 b^{2\alpha+1} d_q b + \sum_{j=m}^\infty \int_0^\infty |\Psi_{q,g}^P(f)(j,b)|^2 b^{2\alpha+1} d_q b. \tag{5.14}$$

(2) **Parseval formula for $\Theta_{q,G}^P$ using $\Psi_{q,g}^P$**
For $f_1, f_2 \in L_{\alpha,q}^2(\mathbb{R}_{q,+})$, we have for all $m \in \mathbb{Z}$,

$$\int_0^\infty f_1(x)\overline{f}_2(x) x^{2\alpha+1} d_q x = \int_0^\infty \Theta_{q,G}^P(f_1)(m,b)\overline{\Theta_{q,G}^P(f_2)(m,b)} b^{2\alpha+1} d_q b + \sum_{j=m}^\infty \int_0^\infty \Psi_{q,g}^P(f_1)(j,b)\overline{\Psi_{q,g}^P(f_2)(j,b)} b^{2\alpha+1} d_q b.$$

Proof
(1) On the one hand, from the relations (5.13) and (5.2), we have for all $m \in \mathbb{Z}$,

$$\int_0^\infty |\Theta_{q,G}^P(f)(m,b)|^2 b^{2\alpha+1} d_q b = \int_0^\infty |\mathcal{F}_{\alpha,q}(f)(x)|^2 \left(\sum_{j=-\infty}^{m-1} \left[\mathcal{F}_{\alpha,q}(g_j^P)(x)\right]^2 \right) x^{2\alpha+1} d_q x.$$

On the other hand, using the relations (3.15) and (4.4), and the Fubini's theorem, we obtain

$$\sum_{j=m}^\infty \int_0^\infty |\Psi_{q,g}^P(f)(j,b)|^2 b^{2\alpha+1} d_q b = \int_0^\infty |\mathcal{F}_{\alpha,q}(f)(x)|^2 \left(\sum_{j=m}^\infty \left[\mathcal{F}_{\alpha,q}(g_j^P)(x)\right]^2 \right) x^{2\alpha+1} d_q x.$$

Hence,

$$\int_0^\infty |\Theta_{q,G}^P(f)(m,b)|^2 b^{2\alpha+1} d_q b + \sum_{j=m}^\infty \int_0^\infty |\Psi_{q,g}^P(f)(j,b)|^2 b^{2\alpha+1} d_q b = \int_0^\infty |\mathcal{F}_{\alpha,q}(f)(x)|^2 \left(\sum_{j=-\infty}^\infty [\mathcal{F}_{\alpha,q}(g_j^P)(x)]^2 \right) x^{2\alpha+1} d_q x.$$

The result follows then from Proposition 9 and Theorem 2.
(2) The assertion (2) follows from (1). □

Theorem 7 *For $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, we have the following reconstruction formulas.*
(1) For all $x \in \mathbb{R}_{q,+}$,

$$f(x) = c_{\alpha,q} \lim_{m\to+\infty} \int_0^\infty \Theta^P_{q,G}(f)(m,b) G^P_{m,b}(x) b^{2\alpha+1} d_q b. \tag{5.15}$$

(2) For all $x \in \mathbb{R}_{q,+}$ and all $m \in \mathbb{Z}$,

$$f(x) = c_{\alpha,q} \int_0^\infty \Theta^P_{q,G}(f)(m,b) G^P_{m,b}(x) b^{2\alpha+1} d_q b + c_{\alpha,q} \sum_{j=m}^{\infty} \int_0^\infty \Psi^P_{q,g}(f)(j,b) g^P_{j,b}(x) b^{2\alpha+1} d_q b. \tag{5.16}$$

Proof
(1) Let $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, fix $x \in \mathbb{R}_{q,+}$ and put $h = \delta_x$. By using the relation (5.12), we get

$$\begin{aligned}(1-q)x^{2\alpha+2} f(x) &= \lim_{m\to+\infty} \int_0^\infty \Theta^P_{q,G}(f)(m,b) \overline{\Theta^P_{q,G}(h)(m,b)} b^{2\alpha+1} d_q b \\ &= \lim_{m\to+\infty} c_{\alpha,q} \int_0^\infty \Theta^P_{q,G}(f)(m,b) \left(\int_0^\infty \overline{h}(t) G^P_{m,b}(t) d_q t \right) b^{2\alpha+1} d_q b \\ &= \lim_{m\to+\infty} c_{\alpha,q} (1-q) x^{2\alpha+2} \int_0^\infty \Theta^P_{q,G}(f)(m,b) G^P_{m,b}(x) b^{2\alpha+1} d_q b.\end{aligned}$$

Thus,

$$f(x) = c_{\alpha,q} \lim_{m\to+\infty} \int_0^\infty \Theta^P_{q,G}(f)(m,b) G^P_{m,b}(x) b^{2\alpha+1} d_q b.$$

(2) The technique of the proof is similar to (1). □

Proposition 16 *For $f \in L^2_{\alpha,q}(\mathbb{R}_{q,+})$, one has for all $j \in \mathbb{Z}$,*

$$\int_0^\infty \Psi^P_{q,g}(f)(j,b) g^P_{j,b}(x) b^{2\alpha+1} d_q b = \int_0^\infty \Theta^P_{q,G}(f)(j+1,b) G^P_{j+1,b}(x) b^{2\alpha+1} d_q b - \int_0^\infty \Theta^P_{q,G}(f)(j,b) G^P_{j,b}(x) b^{2\alpha+1} d_q b.$$

Proof
Using the relations (5.16) and (5.6), and Theorem 2, we obtain

$$\begin{aligned}&\int_0^\infty \Theta^P_{q,G}(f)(j+1,b) G^P_{j+1,b}(x) b^{2\alpha+1} d_q b - \int_0^\infty \Theta^P_{q,G}(f)(j,b) G^P_{j,b}(x) b^{2\alpha+1} d_q b \\ &= \int_0^\infty \mathcal{F}_{\alpha,q}[\mathcal{F}_{\alpha,q}(f *_B \overline{G^P_{j+1}})](-b) G^P_{j+1,b}(x) b^{2\alpha+1} d_q b - \int_0^\infty \mathcal{F}_{\alpha,q}[\mathcal{F}_{\alpha,q}(f *_B \overline{G^P_j})](-b) G^P_{j,b}(x) b^{2\alpha+1} d_q b \\ &= \int_0^\infty \mathcal{F}_{\alpha,q}(f)(b) \left(\left[\mathcal{F}_{\alpha,q}(G^P_{j+1})\right]^2 - \left[\mathcal{F}_{\alpha,q}(G^P_j)\right]^2 \right)(b) j_\alpha(bx;q^2) b^{2\alpha+1} d_q b \\ &= \int_0^\infty \mathcal{F}_{\alpha,q}(f)(b) \left[\mathcal{F}_{\alpha,q}(g^P_j)\right]^2 (b) j_\alpha(bx;q^2) b^{2\alpha+1} d_q b \\ &= \int_0^\infty \Psi^P_{q,g}(f)(j,b) g^P_{j,b}(x) b^{2\alpha+1} d_q b.\end{aligned}$$

□

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