Advances in Analysis, Vol. 1, No. 2, October 2016
https://dx.doi.org/10.22606/aan.2016.12003

Pulse Vaccination Strategy in an SIVS Epidemic Model with General Nonlinear Incidence Rate

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Abstract In this paper, a mathematical model for $SIVS$ including impulsive vaccination strategy and general incidence is proposed and analyzed. By applying the comparison theorem, scaling method techniques for inequalities and stroboscopic map, the sufficient conditions which guarantee the globally attractive of the disease-free periodic solution and the permanence of the disease are established, that is, the disease dies out if $R_1 < 1$, whereas the disease persists if $R_2 > 1$.

Keywords: $SIVS$ epidemic model; pulse control; general nonlinear incidence rate.

1 Introduction

During the past decades, infectious diseases brought huge disaster to the development and survival of human and animal species. So control of infectious diseases has always been a major issue in today's world. In recent years, with Hepatitis B, one after another outbreak of bird flu, AIDS and other diseases, the method which uses dynamics to study the pathogenesis of infectious diseases, the law of propagation and the importance of the control strategy is increasingly prominent. So many scholars have also established a large number of dynamics systems which can reflect the characteristics of infectious diseases , and obtained many important results.

Vaccination is a vital measure in disease control. The standard conventional approach has been constant vaccination, or a uniform and continuous effort of administering the vaccine to a population to try to stop the outbreak. Pulse vaccination works by repeatedly applying a vaccine over defined age ranges (see [1]), and at each vaccination time, a constant fraction of the susceptible population who will lose temporary or permanent acquired immunity is vaccinated. Recently, many authors have studied the pulse vaccination effects on controlling the spread of the diseases (see [2]-[8]).

Several epidemiological models have studied the effect of seasonal transmission on the behavior of diseases. Incorporating pulse vaccination strategy (PVS) and seasonal transmission into epidemiological models, d'Onofrio (see [9]) analyzes the effects of PVS in SIR and SEIR epidemic models ruled by nonlinear force of infection. To research the spread process of infectious diseases and the dynamic behavior by means of mathematical model, the most critical part is the description of the incidence rate (see [10]-[13]). Bilinear incidence rate βSI and standard incidence rate $\beta SI/N$, have been widely used in the classical epidemiology models (see [14]-[16]). With the deep understanding of the model of the spread of disease and the statistical data analysis of the epidemic scholar, the study of the change rule of disease by nonlinear contact rate tally better with the real cases and the results of the model is more general. Then a variety of nonlinear incidence rates have been frequently used in epidemic models (see [17]-[19]). In recent years, it's found that there are some different nonlinear incidence rates in the studies, such as $\beta S^q I$ (see [10]), $\frac{kI^p S}{1+\alpha I^p}$ (see [10],[11]), $\frac{\beta SI}{1+\alpha S+bI}$ (see [12]), $\frac{\beta(t)SI}{1+\alpha S}$ (see [20]), and so on. Motivated by these facts, in this paper we formulate a mathematical model with general periodic variation nonlinear incidence and time-varying pulse control.

This paper is organized as follows. In the next section, we mainly investigate a mathematical model with general periodic variation nonlinear incidence and time-varying pulse control, under some assumptions and the biological interpretation. We also give some useful lemmas which will be used to prove our results. In Section 3, we show that the global attractivity of the disease-free periodic solution is determined by the threshold parameter R_1. In Section 4, we give another expression of threshold parameter R_2, and show that if $R_2 > 1$, the disease is permanent. A brief discussion is given in the last section.

2 Model and Preliminaries

We assume that a population is partitioned into three groups: susceptible, infectious and immune individual. $S(t), I(t), V(t)$ denote the fractions of the population that are susceptible, infectious and immune individual at time t, respectively. In this paper, we formulate a mathematical model with general periodic variation nonlinear incidence and time-varying pulse control. The system is modeled by the following equations:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dfrac{dS(t)}{dt} = \mu - f(t,S,I) - \mu S(t) + \rho I(t) + rV(t), \\ \dfrac{dI(t)}{dt} = f(t,S,I) + g(t,V,I) - \rho I(t) - \mu I(t), \\ \dfrac{dV(t)}{dt} = -g(t,V,I) - (\mu + r)V(t), \end{array}\right\} \quad t \neq t_k, \\ \left.\begin{array}{l} S(t^+) = (1-\theta_k)S(t), \\ I(t^+) = I(t), \\ V(t^+) = V(t) + \theta_k S(t), \end{array}\right\} \quad t = t_k, (k \in N), \end{array}\right. \tag{1}$$

where all coefficients are positive constants. The model is derived from the following assumptions.

(H_1) $S(t), I(t)$ and $V(t)$ are left continuous for $[t_0, +\infty)$, that is, $S(t) = \lim\limits_{h\to 0^+} S(t-h)$, $I(t) = \lim\limits_{h\to 0^+} I(t-h)$ and $V(t) = \lim\limits_{h\to 0^+} V(t-h)$.

(H_2) $\mu > 0$ denotes the natural death rate of susceptible and infective and immunity individual; $\rho > 0$ is the natural recovery rate from the infected; $r \geq 0$ is the rate at which recovered individuals lose immunity and return to the susceptible class.

(H_3) There exist positive integer q and positive ω such that $t_{k+q} = t_k + \omega$ for all $k \in N$. And $\theta_k (0 \leq \theta_k < 1)$ is the inoculation rate of susceptible proportion at each fixed time $t = t_k$, and $\theta_{k+q} = \theta_k$.

(H_4) The general nonlinear incidence rate $f(t,S,I)$ and $g(t,V,I)$ are both piecewise continuous and positive ω-periodic functions. The form of $f(t,S,I)$ and $g(t,V,I)$ is as follows:

$$f(t,S,I) = \begin{cases} f_1(t,S,I), & t \in (n\omega + t_0, n\omega + t_1], \\ \quad \vdots & \\ f_q(t,S,I), & t \in (n\omega + t_{q-1}, n\omega + t_q], \end{cases}$$

$$g(t,V,I) = \begin{cases} g_1(t,V,I), & t \in (n\omega + t_0, n\omega + t_1], \\ \quad \vdots & \\ g_q(t,V,I), & t \in (n\omega + t_{q-1}, n\omega + t_q], \end{cases}$$

for all integer $n \geq 0$, and $f_l(t,0,I) = f_l(t,S,0) = 0$, $g_l(t,0,I) = g_l(t,V,0) = 0$ for $l = 1, 2, \cdots, q$.

Noting that $V(t) = 1 - S(t) - I(t)$, the dynamics of model (1) are determined by the following system

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dfrac{dS(t)}{dt} = \mu - f(t,S,I) - \mu S(t) + \rho I(t) + r(1 - S(t) - I(t)), \\ \dfrac{dI(t)}{dt} = f(t,S,I) + g(t,1-S-I,I) - \rho I(t) - \mu I(t), \end{array}\right\} \quad t \neq t_k, \\ \left.\begin{array}{l} S(t^+) = (1-\theta_k)S(t), \\ I(t^+) = I(t), \end{array}\right\} \quad t = t_k, (k \in N). \end{array}\right. \tag{2}$$

Lemma 1. Let us consider the following impulsive differential equations:

$$\begin{cases} \dfrac{dx(t)}{dt} = a - bx(t),\ t \neq t_k \\ x(t^+) = (1-\theta_k)x(t),\ t = t_k, \end{cases} \tag{3}$$

where $a > 0, b > 0, 0 \leq \theta_k \leq 1$ and there is a positive integer q such that $\theta_{k+q} = \theta_k, t_{k+q} = t_k + \omega$. Then there exists a unique positive periodic solution of system (3)

$$\begin{aligned} x^*(t) = \frac{a}{b} + \Bigg[\left(\prod_{i=1}^{l-1} (1-\theta_i) \right) e^{-b(t_{l-1}-t_0)} x_0^* \\ + \frac{a}{b} \sum_{j=1}^{l-1} \prod_{i=j}^{l-1} (1-\theta_i) e^{-bt_{l-1}} \left(e^{bt_i} - e^{bt_{i-1}} \right) - \frac{a}{b} \Bigg] e^{-b(t-n\omega-t_{l-1})}, \\ n\omega + t_{l-1} < t \leq n\omega + t_l,\ l = 1, 2, \cdots, q,\ n \in \mathbb{N}, \end{aligned} \tag{4}$$

which is globally asymptotically stable, where

$$x_0^* = x(t_0^+) = \frac{\dfrac{a}{b} \sum\limits_{j=1}^{q} \prod\limits_{i=j}^{q} (1-\theta_i) e^{-bt_q} \left(e^{bt_i} - e^{bt_{i-1}} \right)}{1 - \prod\limits_{i=1}^{q} (1-\theta_i) e^{-b\omega}}.$$

Proof. Integrating and solving the first equation of system (3) between pulses for $n\omega + t_{k-1} < t \leq n\omega + t_k, k = 1, 2, \cdots, q, n \in \mathbb{N}$,

$$x(t) = \frac{a}{b} + \left(x(n\omega + t_{k-1}^+) - \frac{a}{b} \right) e^{-b(t-n\omega-t_{k-1})},$$

where $x(n\omega + t_{k-1}^+) = (1-\theta_{k-1})x(n\omega + t_{k-1})$ and $x(n\omega + t_{k-1})$ is the initial value at time $n\omega + t_{k-1}$. Using the second equation of system (2.3), we have

$$x(n\omega + t_1^+) = (1-\theta_1)e^{-b(t_1-t_0)} x(n\omega + t_0^+) + \frac{a}{b} \left(1 - e^{-b(t_1-t_0)} \right) (1-\theta_1),$$

$$\begin{aligned} x(n\omega + t_2^+) = (1-\theta_2)(1-\theta_1)e^{-b(t_2-t_0)} x(n\omega + t_0^+) \\ + \frac{a}{b} \left(e^{-b(t_2-t_1)} - e^{-b(t_2-t_0)} \right) (1-\theta_2)(1-\theta_1) + \frac{a}{b} \left(1 - e^{-b(t_2-t_1)} \right) (1-\theta_2). \end{aligned}$$

Using the inductive method, we have

$$\begin{aligned} x(n\omega + t_q^+) = x\left((n+1)\omega + t_0^+ \right) = \left(\prod_{i=1}^{q} (1-\theta_i) \right) e^{-b(t_q-t_0)} x(n\omega + t_0^+) \\ + \frac{a}{b} \sum_{j=1}^{q} \prod_{i=j}^{q} (1-\theta_i) e^{-bt_q} (e^{bt_i} - e^{bt_{i-1}}). \end{aligned} \tag{5}$$

Set $X_n = x(n\omega + t_0^+)$. From (5) and $t_q - t_0 = \omega$, we have

$$X_{n+1} = \left(\prod_{i=1}^{q} (1-\theta_i) \right) e^{-b\omega} X_n + \frac{a}{b} \sum_{j=1}^{q} \prod_{i=j}^{q} (1-\theta_i) e^{-bt_q} (e^{bt_i} - e^{bt_{i-1}}) \triangleq f(X_n), \tag{6}$$

where f is the stroboscopic map. It is easy to see that system (6) has a unique positive equilibrium

$$x_0^* = x(t_0^+) = \frac{\frac{a}{b}\sum\limits_{j=1}^{q}\prod\limits_{i=j}^{q}(1-\theta_i)e^{-bt_q}\left(e^{bt_i}-e^{bt_{i-1}}\right)}{1-\prod\limits_{i=1}^{q}(1-\theta_i)e^{-b\omega}}.$$

Since $f(X)$ is a straight line with slope less than 1, we obtain that x_0^* is globally asymptotically stable. It implies that the corresponding periodic solution of system (3) $x^*(t)$ is globally asymptotically stable. The proof of Lemma 1 is completed. □

3 Global Attractivity of the Disease-free Periodic Solution

Let us firstly demonstrate the existence of the infection-free periodic solution of system (2). Clearly, if $I(0) = 0$, then $I(t) \equiv 0$ for all $t \geq 0$. Under this condition, the growth of susceptible individuals must satisfy:

$$\begin{cases} \dfrac{dS(t)}{dt} = (\mu + r) - (\mu + r)S(t),\ t \neq t_k, \\ S(t^+) = (1-\theta_k)S(t),\ t = t_k. \end{cases} \tag{7}$$

According to Lemma 1, we know that the periodic solution of system (7)

$$\begin{aligned} S^*(t) = 1 + \Bigg[&\left(\prod_{i=1}^{l-1}(1-\theta_i)\right) e^{-(\mu+r)(t_{l-1}-t_0)}S_0^* \\ &+ \sum_{j=1}^{l-1}\prod_{i=j}^{l-1}(1-\theta_i)e^{-(\mu+r)t_{l-1}}\left(e^{(\mu+r)t_i}-e^{(\mu+r)t_{i-1}}\right) - 1\Bigg] e^{-(\mu+r)(t-n\omega-t_{l-1})}, \\ & n\omega + t_{l-1} < t \leq n\omega + t_l,\ l = 1, 2, \cdots, q, n \in \mathbb{N}, \end{aligned} \tag{8}$$

which is globally asymptotically stable, where

$$S_0^* = S(t_0^+) = \frac{\sum\limits_{j=1}^{q}\prod\limits_{i=j}^{q}(1-\theta_i)e^{-(\mu+r)t_q}\left(e^{(\mu+r)t_i}-e^{(\mu+r)t_{i-1}}\right)}{1-\prod\limits_{i=1}^{q}(1-\theta_i)e^{-(\mu+r)\omega}}.$$

From system (2), we can easily obtain that the solution $(S^*(t), 0)$ is the disease-free periodic solution. To discuss the attractivity of the disease-free periodic solution of system (2), we firstly assume the following hypothesis.

(**A**): There exist two piecewise continuous and positive ω-periodic functions $\beta_l(t), \gamma_l(t)$, that is $\gamma_l(t) = \gamma_l(t + n\omega), \beta_l(t) = \beta_l(t + n\omega)$, for all $l = 1, 2, \cdots, q$ and $n \in N$, such that $f_l(t, S, I) \leq \beta_l(t)S(t)I(t), g_l(t, V, I) \leq \gamma_l(t)V(t)I(t)$ and $t \geq t_0$.

Theorem 1. If $R_1 < 1$ and system (2) satisfies the hypothesis (A), then the disease-free periodic solution $(S^*(t), 0)$ of system (2) is globally asymptotically stable, where

$$R_1 = \frac{\sum\limits_{i=1}^{q}\int_{t_{i-1}}^{t_i}\varphi_i(t)\,dt}{\omega(\mu+\rho)}. \tag{9}$$

$$\varphi_l(t) = \max\{\gamma_l(t), \beta_l(t)\},\ l = 1, 2, \cdots, q.$$

Proof. Since $R_1 < 1$, we have

$$\Lambda \triangleq \exp\left[\sum_{i=1}^{q}\int_{t_{i-1}}^{t_i}\varphi_i(t)\,dt - \omega(\mu+\rho)\right] < 1.$$

From the second equation of system (1), for $t \in (n\omega + t_{l-1}, n\omega + t_l]$ $(l = 1, 2, \cdots, q)$,

$$\begin{aligned}\frac{dI(t)}{dt} &= f(t,S,I) + g(t,V,I) - \mu I(t) - \rho I(t)\\ &\le \beta_l(t)S(t)I(t) + \gamma_l(t)V(t)I(t) - (\mu+\rho)I(t)\\ &\le \varphi_l(t)(S(t)+V(t))I(t) - (\mu+\rho)I(t)\\ &\le [\varphi_l(t) - (\mu+\rho)]\,I(t).\end{aligned}$$

Then, we obtain

$$I(t) \le I(n\omega + t_{l-1})\exp\left(\int_{n\omega+t_{l-1}}^{t}[\varphi_l(\tau) - (\mu+\rho)]\,d\tau\right).$$

By using the similar method, we can infer that for $t \in (n\omega + t_{l-1}, n\omega + t_l]$,

$$I(t) \le I(n\omega+t_0)\exp\left[\int_{n\omega+t_0}^{n\omega+t_1}\varphi_1(\tau)d\tau + \cdots + \int_{n\omega+t_{l-1}}^{t}\varphi_l(\tau)d\tau - (\mu+\rho)(t-n\omega-t_0)\right].$$

Especially, when $t = (n+1)\omega + t_0$, we have

$$\begin{aligned}I[(n+1)\omega+t_0] = I(n\omega+t_q) &\le I(n\omega+t_0)\exp\left[\sum_{i=1}^{q}\int_{t_{i-1}}^{t_i}\varphi_i(\tau)d\tau - (\mu+\rho)\omega\right]\\ &= \Lambda I(n\omega+t_0).\end{aligned} \tag{10}$$

Therefore, for any position integers, we have $I[(n+s)\omega+t_0)] \le \Lambda^s I(n\omega+t_0)$. It follows from (10) that

$$I[(n+s)\omega+t_0)] \le \Lambda^s I(n\omega+t_0) \to 0, \text{ as } s \to \infty. \tag{11}$$

From (11), we get

$$\lim_{t\to\infty} I(t) = 0. \tag{12}$$

Therefore, for any arbitrarily small $\varepsilon > 0$, there exists $t^1 (> t_0)$, such that

$$I(t) < \varepsilon, \text{ for all } t > t^1.$$

From the equation of system (2), we have for $t > t^1$,

$$\begin{aligned}\frac{dS(t)}{dt} &= \mu - f(t,S,I) - \mu S(t) + \rho I(t) + r(1 - S(t) - I(t))\\ &\ge (\mu+r) - \beta_k(t)S(t)I(t) - (\mu+r)S(t) - rI(t)\\ &\ge (\mu + r - r\varepsilon) - (\beta^*\varepsilon + \mu + r)S(t),\end{aligned}$$

where $\beta^* = \max\{\beta_k(t),\ t_0 \le t \le t_0+\omega,\ k = 1, \cdots, q\}$,
and

$$\begin{aligned}\frac{dS(t)}{dt} &= \mu - f(t,S,I) - \mu S(t) + \rho I(t) + r(1 - S(t) - I(t))\\ &\le (\mu + r + \rho\varepsilon) - (\mu+r)S(t),\end{aligned}$$

where ε is arbitrarily small. Therefore, let $\varepsilon \to 0$, we can easily obtain that

$$\lim_{t\to\infty}(S(t) - S^*(t)) = 0. \tag{13}$$

From (12) and (13), we can see that the disease-free periodic solution $(S^*(t), 0)$ is globally attractive. □

4 Permanence of the Disease

In this section, we mainly obtain the sufficient condition for the permanence of system (2). At first, we give the following hypothesis (B).

(**B**): There exists a piecewise continuous and positive ω-periodic functions $\alpha_l(t)$, such that $f_l(t,S,I) \geq \alpha_l(t)S(t)I(t)$ for $t \geq t_0$.

Theorem 2. If $R_2 > 1$ and system (2) satisfies the hypotheses (A) and (B), then system (2) is uniformly persistent, where

$$R_2 = \frac{\sum\limits_{i=1}^{q} \int_{t_{i-1}}^{t_i} S^*(\tau)\alpha_i(\tau)d\tau}{(\mu+\rho)\omega}, \tag{14}$$

where $S^*(t)$ is defined in (8).

Proof. Since $R_2 > 1$, we can easily see that there exists a sufficiently small $\epsilon > 0$ such that

$$\Omega \triangleq \exp\left[\sum_{i=1}^{q}\int_{t_{i-1}}^{t_i}(y^*(\tau)-\epsilon)\alpha_i(\tau)d\tau - (\mu+\rho)\omega\right] > 1, \tag{15}$$

where $y^*(t)$ is the periodic solution of system (16).

In order to illustrate the conclusion, we firstly prove that the disease is uniformly weakly persistent, that is, there exists a positive constant $\eta > 0$, such that $\limsup\limits_{t\to+\infty} I(t) \geq \eta$. By contradiction, for above $\epsilon > 0$, we have that there exists $t^2 > t_0$, such that $I(t) < \epsilon$ for all $t > t^2$.

In view of the hypothesis (A) and the first equation of system (2), for all $t > t^2$, we have

$$\begin{aligned}\frac{dS(t)}{dt} &= \mu - f(t,S,I) - \mu S(t) + \rho I(t) + r(1-S(t)-I(t))\\ &\geq (\mu+r) - \beta_k(t)S(t)I(t) - (\mu+r)S(t) - rI(t)\\ &\geq (\mu+r-r\epsilon) - (\beta^*\epsilon+\mu+r)S(t),\end{aligned}$$

where $\beta^* = \max\{\beta_k(t),\ t_0 \leq t \leq t_0+\omega,\ k = 1,\cdots,q\}$.

Consider the following auxiliary impulsive system:

$$\begin{cases}\dfrac{dy(t)}{dt} = (\mu+r-r\epsilon) - (\beta^*\epsilon+\mu+r)y(t),\ t \neq t_k,\\ y(t^+) = (1-\theta_k)y(t),\ t = t_k,\\ y(t_0^+) = S_0 > 0.\end{cases} \tag{16}$$

By Lemma 1, system (16) has a unique periodic solution $y^*(t)$, which is globally asymptotically stable.

By comparison theorem, we have $S(t) \geq y(t)$ and $y(t) \to y^*(t)$ as $t \to +\infty$, where $y(t)$ is the solution of the comparison system (16).

Therefore, for the above mentioned $\epsilon > 0$, there exists a $t' > 0$, such that

$$S(t) \geq y(t) \geq y^*(t) - \epsilon \quad \text{for all } t > t^2 + t'. \tag{17}$$

For the above mentioned $t^2 + t'$, we know that there exists a position integer n_1 such that $n_1\omega \geq t^2 + t'$. Then, for all $n\omega + t_{k-1} < t \leq n\omega + t_k (n \geq n_1, k = 1,\cdots,q)$, from (17) and the second equation of system (2), we have

$$\begin{aligned}\frac{dI(t)}{dt} &= f(t,S,I) + g(t,V,I) - (\mu+\rho)I(t)\\ &\geq \alpha_k(t)S(t)I(t) - (\mu+\rho)I(t)\\ &\geq \left[\alpha_k(t)(y^*(t)-\epsilon) - (\mu+\rho)\right]I(t).\end{aligned} \tag{18}$$

Then, we obtain that for $t \geq n_1\omega + t_0$

$$I(t) \geq I(n\omega + t_{k-1}) \exp\left(\int_{n\omega+t_{k-1}}^{t} [(y^*(\tau) - \epsilon)\alpha_k(\tau) - (\mu + \rho)]\, d\tau\right).$$

By using the similar method, we can infer that for $t \in (n\omega + t_{k-1}, n\omega + t_k]$ (here $n \geq n_1$),

$$\begin{aligned} I(t) \geq I(n\omega + t_0) \exp\Bigg(&\int_{n\omega+t_0}^{n\omega+t_1} (y^*(\tau) - \epsilon)\alpha_1(\tau)d\tau + \cdots \\ &+ \int_{n\omega+t_{k-1}}^{t} (y^*(\tau) - \epsilon)\alpha_k(\tau)d\tau - (\mu + \rho)(t - n\omega - t_0)\Bigg]. \end{aligned}$$

Especially, when $t = (n + 1)\omega + t_0$, we have

$$\begin{aligned} I[(n+1)\omega + t_0] = I(n\omega + t_q) &\geq I(n\omega + t_0) \exp\left[\sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} (y^*(\tau) - \epsilon)\alpha_i(\tau)d\tau - (\mu + \rho)(t_q - t_0)\right] \\ &= I(n\omega + t_0) \exp\left[\sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} (y^*(\tau) - \epsilon)\alpha_i(\tau)d\tau - (\mu + \rho)\omega\right] \\ &= \Omega I(n\omega + t_0). \end{aligned} \tag{19}$$

Therefore, it follows from (19) that, for any position integer s, $I[(n + s)\omega + t_0] \geq \Omega^s I(n\omega + t_0)$. By (15), we have

$$I[(n + s)\omega + t_0)] \geq \Omega^s I(n\omega + t_0) \to \infty, \text{ as } s \to \infty. \tag{20}$$

From the above discussion, we can get $\lim\limits_{t\to\infty} I(t) = \infty$, which is a contradiction to $0 < I(t) < \epsilon$. Thus the claim is proved, that is, there is a $\sigma > 0$ such that $\limsup\limits_{t\to\infty} I(t) \geq \sigma$.

By the claim, we are left to consider the following two possibilities:

Case 1. $I(t) > \sigma$ for t large enough.

Case 2. $I(t)$ oscillates about σ for t large enough.

The conclusion is evident in the first case. For the second case, set $m = \min\{\frac{\sigma}{2}, m^*\}$, where $m^* = \min\{\sigma \exp[-(\mu + \rho)\omega],\ \sigma \exp[-(\mu + \rho)\omega + h\omega]\}$ and $h = \max\{\alpha_k(s)(y^*(s) - \epsilon) - (\mu + \rho),\ t_0 \leq s \leq t_0 + \omega,,\ k = 1, 2, \cdots, q\}$.

We hope to show that $I(t) \geq m$ for t large enough.

Since $I(t)$ oscillates about σ eventually. Let $t^* > 0$ and $\xi > 0$ satisfy

$I(t^*) = I(t^* + \xi) = \sigma$, and $I(t) < \sigma$ for $t^* < t < t^* + \xi$,

where t^* is large enough such that $S(t) \geq y^*(t) - \epsilon$ for $t^* < t < t^* + \xi$.

$I(t)$ is not effected by impulses and $I(t)$ is uniformly continuous. Hence, there is a λ ($0 < \lambda < \omega$, and λ is independent of the choice of t^*) such that $I(t) > \frac{\sigma}{2} \geq m$ for $t^* < t < t^* + \lambda$. If $\xi \leq \lambda$, there is nothing to prove. Let us consider the case $\xi > \lambda$, there are two possible cases for ξ.

(a) If $\lambda < \xi \leq \omega$, then from system (2), we have

$$\begin{aligned} \frac{dI(t)}{dt} &= f(t, S, I) + g(t, V, I) - (\mu + \rho)I(t) \\ &\geq -(\mu + \rho)I(t). \end{aligned} \tag{21}$$

It follows from (21) and $I(t^*) = \sigma$, that

$$I(t) \geq I(t^*) \exp[-(\mu + \rho)(t - t^*)] \geq \sigma \exp[-(\mu + \rho)(t^* + \lambda - t^*)] \geq \sigma \exp[-(\mu + \rho)\omega] \geq m.$$

Then $I(t) \geq m$ for all $t \in [t^*, t^* + \xi]$.

(b) If $\xi > \omega$, then from the discussion in subcase (a), we have $I(t^* + \omega) \geq \sigma \exp[-(\mu + \rho)\omega]$. For any $t > t^* + \omega$, we can choose an integer $l \geq 0$ such that $t \in [t^* + \omega + l\omega,\ t^* + \omega + (l + 1)\omega)$. Integrating the

inequality (19) from $t^* + \omega$ to t, it follows from (15) that

$$\begin{aligned} I(t) &\geq I(t^* + \omega) \exp\left(\int_{t^*+\omega}^{t} [\alpha_k(s)(y^*(s) - \epsilon) - (\mu + \rho)]ds\right) \\ &\geq \sigma \exp[-(\mu + \rho)\omega] \exp\left\{\int_{t^*+\omega}^{t^*+(l+1)\omega} + \int_{t^*+(l+1)\omega}^{t} [\alpha_k(s)(y^*(s) - \epsilon) - (\mu + \rho)]ds\right. \\ &\geq \sigma \exp[-(\mu + \rho)\omega] \Omega^l \exp\left(\int_{t^*+(l+1)\omega}^{t} [\alpha_k(s)(y^*(s) - \epsilon) - (\mu + \rho)]ds\right) \\ &\geq \sigma \exp[-(\mu + \rho)\omega + h\omega] \\ &\geq m. \end{aligned}$$

Thus we have that $I(t) \geq m$ for $t \in [t^*,\ t^* + \xi]$. Since this kind of interval $[t^*, t^* + \xi]$ is chosen in an arbitrary way (we only need t^* to be large enough), then we conclude that $I(t) \geq m$ for t large enough.

According to our above discussion, the choice of m is independent of the positive solution of system (2), and we have proved that any solution of system (2) satisfies $I(t) \geq m$ for sufficiently large t, that is, $\liminf_{t\to+\infty} I(t) \geq m$. It is easy to obtain that, there exists a positive constant S_* such that $\liminf_{t\to+\infty} S(t) \geq S_*$. Therefore, system (2) is permanent. □

5 Conclusion

In this paper, we have formulated a mathematical model with general periodic variation nonlinear incidence and time-varying pulse control. Under the reasonable assumptions, we have showed that if the hypothesis (A) and $R_1 < 1$ hold, the infectious population dies out, and the disease-free periodic solution is globally attractive. If the hypotheses (A) and (B) hold and $R_2 > 1$, the infections population persists. Owing to the application of the method of enlarging and reducing, we can not obtain the basic reproductive number R_0, which determines whether the disease persists or not. This issue would be left as our future consideration.

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