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A Note on Positive Periodic Solutions of the Superlinear Heat Equation with Inhomogeneity

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Abstract This paper deals with the existence of positive periodic solutions of the superlinear heat equation with inhomogeneity. We first prove the existence of positive periodic solutions to the equation with an indispensable nontrivial and globally small inhomogeneity; and then the non-existence for a locally large inhomogeneity with possibly small support is obtained.

Keywords: Positive periodic solutions, locally large inhomogeneity, globally small inhomogeneity.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $f(x,t)$, $m(x,t)$ be two positive functions which are T-periodic with respect to the time t. We are concerned with the existence of positive periodic solutions for the following inhomogeneous superlinear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = m(x,t)u^q + f(x,t), \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,t) = u(x,t+T), \quad (x,t) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{1}$$

where $q > 1$.

The periodic problem (1) with $f \equiv 0$ has been intensively studied by many authors since the last century. It is known that there exists at least one positive periodic solution if $1 < q < (N+2)/(N-2)$ ($q > 1$ for $N = 1, 2$) and no positive periodic solution provided that $q \geq (N+2)/(N-2)$ with $N > 2$ and Ω is strictly star-shaped, see [1,2,6,7,8] and references therein. As for the inhomogeneous case $f \not\equiv 0$, it was Ôtani [5, J. Diff. Equations, 1984] who proved that there exists a periodic solution for (1) if f is sufficiently small with $1 < q < (N+2)/(N-2)$. Further, Hirano and Mizoguchi [3, Proc. Amer. Math. Soc., 1995] showed that there exist a stable and an unstable positive periodic solutions for the problem (1) under similar conditions.

The purpose of the present note is to give a complete characterization for the existence and non-existence of nontrivial periodic solutions for all the possible value of the exponent $q > 1$. As a supplementation, we obtain the existence of the positive periodic solution for the indispensable nontrivial and globally small inhomogeneity $f(x,t)$ with $q \geq (N+2)/(N-2)$, and we also find the non-existence result for the locally large inhomogeneity $f(x,t)$ with possibly small support and $q > 1$.

For simplicity, we put $Q_T = \Omega \times (0,T)$ and let $B_r(x_0)$ be the ball with radius r centered at x_0. Throughout this paper we assume that $q > 1$ and

$$\underline{m} \leq m(x,t) \leq \overline{m}, \quad f(x,t) \geq 0, \quad f(x,t) \not\equiv 0, \qquad (x,t) \in Q_T$$

for some positive constants $\overline{m} \geq \underline{m} > 0$.

We state our main results here.

Theorem 1.1 *For any $x_0 \in \Omega$ and $r > 0$, the problem (1) admits no positive periodic solution provided that $\inf\limits_{(B_r(x_0)\cap\Omega)\times(0,T)} f(x,t)$ is sufficiently large; while, the problem (1) admits at least one positive periodic solution provided that $\sup\limits_{Q_T} f(x,t)$ is sufficiently small.*

2 Proof of the main results

Before proving the non-existence result, we present a lower bound estimate based on the comparison principle with weak lower solutions.

Lemma 2.1 *Suppose that* $u, v \in W_2^{1,1}(Q_T)$ *such that*

$$\frac{\partial u}{\partial t} - \Delta u \geq \frac{\partial v}{\partial t} - \Delta v, \quad (x,t) \in Q_T$$

in the sense of distributions, and u, v *satisfy* $u(x,T) = u(x,0)$, $v(x,T) = v(x,0)$ *for* $x \in \Omega$, $u(x,t) \geq v(x,t)$ *for* $x \in \partial\Omega$, $t \in (0,T)$. *Then* $u(x,t) \geq v(x,t)$ *for* $(x,t) \in Q_T$.

Proof. Let $z = v - u$ and $\varphi = (v-u)_+$. Then $\varphi, z \in W_2^{1,1}$, φ and z are T-periodic with respect to t, $\varphi(x,t) = 0$ for $x \in \partial\Omega$, and

$$z_t - \Delta z \leq 0, \quad (x,t) \in Q_T$$

in the sense of distributions. Taking φ as the test function, we see that

$$\iint_{Q_T} (\varphi\varphi_t + |\nabla\varphi|^2) \mathrm{d}x\mathrm{d}t \leq 0.$$

Combining with the periodicity and the Poincaré inequality, we have $\varphi = 0$ a.e. Q_T. □

Lemma 2.2 *Assume that* u *is the solution of the following problem*

$$\begin{cases} -\Delta u = g, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $g(x) \geq 0$ *for* $x \in \Omega$, *and* $g(x) \geq K > 0$ *for* $x \in B_r(x_0) \subset \Omega$. *Then* $u(x) \geq 0$ *for* $x \in \Omega$, *and* $u(x) \geq \frac{K}{4N} r^2$ *for* $x \in B_{r/\sqrt{2}}(x_0)$.

Proof. Let

$$v(x) = \frac{K}{2N}(r^2 - |x - x_0|^2)_+.$$

Then $v \in H_0^1(\Omega)$ and

$$-\Delta v \leq K\chi_{B_r(x_0)} \leq g(x), \quad x \in \Omega$$

in the sense of distributions, where $\chi_{B_r(x_0)}$ is the characteristic function of the ball $B_r(x_0)$. Lemma 2.1 implies that

$$u(x) \geq v(x) \geq \frac{K}{4N} r^2 \chi_{B_{r/\sqrt{2}}(x_0)}.$$

The proof is completed. □

Lemma 2.3 *For any* $q > 1$, $x_0 \in \Omega$ *and* $r > 0$, *there exists a positive constant* $K = K(q, r, x_0)$, *such that if*

$$\inf_{(x,t) \in (B_r(x_0) \cap \Omega) \times (0,T)} f(x,t) \geq K,$$

then the problem (1) *admits no positive periodic solution.*

Proof. Assume that $u(x,t)$ is a positive periodic solution of the problem (1). Since $x_0 \in \Omega$, we can choose $x_0' \in \Omega$ and $r' > 0$ such that $B_{r'}(x_0') \subset \Omega \cap B_r(x_0)$. Without loose of generality, we assume that $B_r(x_0) \subset \Omega$. Let u_1 be the solution of the following problem

$$\begin{cases} -\Delta u_1 = K\chi_{B_r(x_0)}, & x \in \Omega, \\ u_1(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\chi_{B_r(x_0)}$ is the characteristic function of the ball $B_r(x_0)$. Then $u_1 \geq 0$ in Ω and

$$u_1(x) \geq \frac{K}{4N} r^2, \quad x \in B_{\frac{r}{\sqrt{2}}}(x_0)$$

according to Lemma 2.2. Furthermore, we see that

$$u(x,t) \geq u_1(x), \quad (x,t) \in Q_T$$

from Lemma 2.1. Consider the following problem

$$\begin{cases} -\Delta u_2 = \underline{m} u_1^q, & x \in \Omega, \\ u_2(x) = 0, & x \in \partial\Omega. \end{cases}$$

We fix a constant $d > 0$ such that

$$\Big(\frac{d}{2}\Big)^q \geq d, \quad d^2 \geq 2^{N/2+3}. \tag{2}$$

This can be done by choosing d sufficiently large since $q > 1$. And then we choose K sufficiently large such that

$$\underline{m}\Big(\frac{r^2}{4N}\Big)^q K^{q-1} \geq d \tag{3}$$

since $q > 1$. Now, we have

$$\underline{m} u_1^q \geq \underline{m}\Big(\frac{K}{4N} r^2\Big)^q \geq dK, \quad x \in B_{\frac{r}{\sqrt{2}}}(x_0).$$

Lemma 2.2 implies that $u_2 \geq 0$ in Ω and

$$u_2(x) \geq \frac{dK}{4N}\Big(\frac{r}{\sqrt{2}}\Big)^2, \quad x \in B_{\frac{r}{2}}(x_0).$$

Similarly, we see that

$$u(x,t) \geq u_2(x), \quad (x,t) \in Q_T$$

from Lemma 2.1. In general, for $n \geq 2$, we consider the following problem

$$\begin{cases} -\Delta u_{n+1} = \underline{m} u_n^q, & x \in \Omega, \\ u_{n+1}(x) = 0, & x \in \partial\Omega. \end{cases}$$

We prove by induction that

$$u_n(x) \geq \frac{d^{n-1}K}{4N} \frac{r^2}{2^{n-1}}, \quad x \in B_{\frac{r}{2^{n/2}}}(x_0),$$

and

$$u(x,t) \geq u_n(x), \quad (x,t) \in Q_T$$

for $n \in \mathbb{N}$. The cases $n = 1, 2$ have already been proved above. Suppose that for some $i \in \mathbb{N}$ the assertion is true. We can verify that

$$\underline{m} u_i^q \geq \underline{m}\Big(\frac{d^{i-1}K}{4N} \frac{r^2}{2^{i-1}}\Big)^q \geq dK\Big(\frac{d}{2}\Big)^{q(i-1)} \geq d^i K, \quad x \in B_{\frac{r}{2^{n/2}}}(x_0),$$

according to (2) and (3). Applying Lemma 2.2, we see that

$$u_{i+1}(x) \geq \frac{d^i K}{4N}\Big(\frac{r}{2^{n/2}}\Big)^2, \quad x \in B_{\frac{r}{2^{(n+1)/2}}}(x_0).$$

The induction is completed by using the comparison principle Lemma 2.1. Thus,

$$\begin{aligned}\|u\|^2_{L^2(Q_T)} &\geq T\|u_{n+1}\|^2_{L^2(\Omega)}\\ &\geq \omega_N T\Big(\frac{d^nK}{4N}\frac{r^2}{2^n}\Big)^2\Big(\frac{r}{2^{(n+1)/2}}\Big)^N\\ &= \omega_N T\Big(\frac{Kr^2}{4N}\Big)^2\Big(\frac{r}{\sqrt{2}}\Big)^N\Big(\frac{d^2}{2^{N/2+2}}\Big)^n\\ &\geq \omega_N T\Big(\frac{Kr^2}{4N}\Big)^2\Big(\frac{r}{\sqrt{2}}\Big)^N\cdot 2^n\end{aligned}$$

from (2), where ω_N is the measure of $B_1(0)$. Hence we have obtained a contradiction to the fact that $u \in L^2(Q_T)$. □

Finally, we employ the monotone iteration technique to show the existence results.

Lemma 2.4 *For any $q > 1$ and $s > \frac{N+2}{2}$, there exists a positive constant $\varepsilon = \varepsilon(q, s)$, such that if $\|f\|_{L^s(Q_T)} \leq \varepsilon$ then the problem (1) admits at least one positive periodic solution.*

Proof. Let $u_1(x,t)$ be the periodic solution of the following problem

$$\begin{cases}\dfrac{\partial u_1}{\partial t} - \Delta u_1 = f(x,t), & (x,t) \in Q_T,\\ u_1(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T),\\ u_1(x,0) = u_1(x,T), & x \in \Omega.\end{cases}$$

The existence and uniqueness is trivial. Since $f \geq 0$ and $f \not\equiv 0$, we see that u_1 is a positive periodic function. For $n \geq 1$, let u_{n+1} be the positive periodic solution of the following problem

$$\begin{cases}\dfrac{\partial u_{n+1}}{\partial t} - \Delta u_{n+1} = m(x,t)u_n^q + f(x,t), & (x,t) \in Q_T,\\ u_{n+1}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T),\\ u_{n+1}(x,0) = u_{n+1}(x,T), & x \in \Omega.\end{cases}$$

Since $u_1 > 0$, we see that $u_2 \geq u_1$. By the induction, we have $u_{n+1} \geq u_n \geq \cdots \geq u_1 > 0$.

Next we show that the sequence $\{u_n\}$ converges provided that $\|f\|_{L^s(Q_T)}$ is sufficiently small. The classical result in [4] implies that

$$\|u_1\|_{L^\infty(Q_T)} \leq C_0\|f\|_{L^s(Q_T)},$$

where C_0 is a positive constant. We choose $\varepsilon > 0$ sufficiently small such that

$$\overline{m}|Q_T|^{\frac{1}{s}}(2C_0)^q\varepsilon^{q-1} \leq 1$$

since $q > 1$. Now, for $\|f\|_{L^s(Q_T)} \leq \varepsilon$, we prove by induction that

$$\|u_n\|_{L^\infty(Q_T)} \leq 2C_0\|f\|_{L^s(Q_T)}, \quad n \in \mathbb{N}. \tag{4}$$

The case $n = 1$ has been proved above. Assume that for some $i \in \mathbb{N}$ the estimate (4) is true. Therefore, we have

$$\begin{aligned}\|u_{i+1}\|_{L^\infty(Q_T)} &\leq C_0\|mu_i^q + f\|_{L^s(Q_T)}\\ &\leq C_0\overline{m}|Q_T|^{\frac{1}{s}}\|u_i\|^q_{L^\infty(Q_T)} + C_0\|f\|_{L^s(Q_T)}\\ &\leq C_0\overline{m}|Q_T|^{\frac{1}{s}}(2C_0\|f\|_{L^s(Q_T)})^q + C_0\|f\|_{L^s(Q_T)}\\ &\leq 2C_0\|f\|_{L^s(Q_T)}.\end{aligned}$$

By the induction, we see that $\|u_n\|_{L^\infty(Q_T)} \leq 2C_0\|f\|_{L^s(Q_T)} \leq 2C_0\varepsilon$. Thus, the sequence $\{u_n\}$ is monotone with respect to n and bounded. There exists a function u_0 such that

$$\lim_{n\to\infty} u_n(x,t) = u_0(x,t), \quad (x,t) \in Q_T.$$

We can verify that u_0 is a positive periodic solution of the problem (1). □

Proof of Theorem 1.1 The non-existence result is shown in Lemma 2.3. We note that

$$\|f\|_{L^s(Q_T)} \leq |Q_T|^{\frac{1}{s}} \sup_{Q_T} |f|,$$

where $|Q_T|$ is the measure of Q_T. Then the existence result is a simple conclusion of Lemma 2.4. □

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