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On the Exact Inverse Problem of the Calculus of Variations

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Abstract The article is devoted to the problem of whether or not a given system of differential equations is identical with the Euler–Lagrange system of an appropriate variational integral. The actual theories which rest on the Helmholz solvability condition and the local Tonti formula are revised. Quite elementary approach is applied. Then the Helmholz condition turns into an easy matter together with unexpected consequence, the solution of incomplete inverse problem. Since the Tonti formula does not give the economical solution, new direct and even global approach is proposed for the determination of all first–order variational integrals related to the second–order Euler–Lagrange system. It employs the fibered de Rham theory where the multiple–valued (ramified) solutions are included as well. The article is of a certain interest also for nonspecialists.

Keywords: Euler–Lagrange expression; divergence; Helmholz condition; exact inverse problem; de Rham theory.

Informally, the inverse problem of the calculus of variations concerns the indication of hidden extremality principles which undoubtedly belongs to the most important topics both in theoretical and in applied sciences. Various setings are possible. Roughly, the *absolute inverse problem* is without any restrictions: to decide whether a given system of differential equations is *equivalent* in the broadest possible sense to an appropriate Euler–Lagrange system. Alas, only some rudiments of the absolute calculus of variations exist [1, Section 7] and partly [2], [3]. Then the *general inverse problem* deals with equivalences preserving the dependent and the independent variables. Only the particular subcase where the equivalences are linear combinations of equations was systematically investigated after the famed initiating article [4], however, the general case with one independent variable was treated in [5]. Finally, the *exact inverse problem* appears as a very strict topic: to determine if a given system of differential equations is *identical* with the Euler–Lagrange system of an appropriate variational integral. This problem looks as the easiest one, however, though it was investigated for a long time in a huge number of articles, the results still cannot be regarded as satisfactory.

Our aim is twofold. First, to demonstrate the simplicity of the well–known fundamental achievements on the exact inverse problem. Second, to propose a direct method of determining the "most economical" solutions of the exact inverse problem for the case of the second–order Euler–Lagrange systems of partial differential equations on rather general domains. This is already a new global approach. It should be noted that very advanced tools were applied to the global inverse problem [6],[7],[8],[9],[10], however, only the abstract existence of smooth and single–valued solutions on the total jet space of all cross–sections of a fibered manifold was ensured if some cohomology of the underlying manifold vanishes. It is nevertheless well–known that just the multiply–valued (ramified) objects with a nontrivial monodromy group and the behaviour near the singularities are regarded for the most interesting in contemporary mathematics and physics.

In more detail, we start with *Preface* where the classical local Helmholz–Tonti theory is made a "childs play" and moreover adapted for the not yet investigated incomplete exact inverse problem [11]. Two letters are enough to express the solvability condition (7) and the proof of the resolving formula (11) occupies not many lines. Alas, these classical results are useless in practice and this provides the reason for the subsequent crucial part *The global strategy.* We deal with the direct construction both of the local and of the global "economical" solutions of the inverse problem. First of all, the introduction of auxiliary functions $G_{ii'}^{jj'}$ is useful even in the local theory: while the correction [6] of Tonti formula provides only the existence of a single special solution, we obtain the overview of all such solutions. In the global theory, we propose the construction even of all ramified solutions. They appear if the equations (30), (31) and (32) are resolved on multiply–valued domains by using de Rham theory which is applied for the first time in this connection. The ramified solutions are parametrized in terms of integrals over cycles representing the homologies. The remaining part of the article includes several short *Particular examples* related to the general theory. Some local aspects of these examples are as a rule treated in large articles. The concluding *Appendix* indicates the reason a little: some seemingly difficult results can be in fact proved on a few lines by an appropriate use of the "naive" methods.

1 Preface: The Local Theory

In order to outline the core of well-known actual achievements, let us introduce the *jet coordinates*

$$x_i,\ w_I^j \quad (i=1,\dots,n;\ j=1,\dots,m;\ I=i_1\cdots i_r;\ |I|=r=0,1,\dots). \tag{1}$$

They are called *independent variables* $x_1,\dots,x_n$, *dependent variables* $w^1,\dots,w^m$ (empty $I=\phi$ with $r=0$) and *higher–order variables* w_I^j (nonempty I with $r\geq 1$) which correspond to derivatives

$$\frac{\partial w^j}{\partial x_I}=\frac{\partial^r w^j}{\partial x_{i_1}\cdots\partial x_{i_r}} \quad (I=i_1\cdots i_r;\ i_1,\dots,i_r=1,\dots,n)$$

in the familiar sense.

We shall deal with C^∞–smooth functions, each depending on a finite number of coordinates (1). However, in this Preface, the functions may depend on a parameter t. So the primary independent variables are completed with the additional term $t\,(=x_{n+1})$ and the higher–order variables w_I^j with the additional *variations* $w_{It}^j, w_{Itt}^j,\dots$ which correspond to derivatives

$$\frac{\partial}{\partial t}\frac{\partial w^j}{\partial x_I}\,(=\frac{\partial}{\partial x_{n+1}}\frac{\partial w^j}{\partial x_I}),\ \frac{\partial^2}{\partial t^2}\frac{\partial w^j}{\partial x_I}\,(=\frac{\partial^2}{\partial x_{n+1}^2}\frac{\partial w^j}{\partial x_I}),\dots.$$

In fact only the *first–order variations* w_{It}^j are important. Altogether we speak of the *extendend jet coordinates.*

Some functions $w^j=w^j(x_1,\dots,x_n,t)$ will be substituted into the functions $F=F(x_1,\dots,x_n,t,\cdot\cdot,w_I^j,w_{It}^j,\cdot\cdot)$ under consideration. Let

$$\begin{aligned}\mathcal{F}&=\mathcal{F}(x_1,\dots,x_n,t)\\ &=F\left(x_1,\dots,x_n,t,\cdot\cdot,\frac{\partial w^j}{\partial x_I}(x_1,\dots,x_n,t),\frac{\partial}{\partial t}\frac{\partial w^j}{\partial x_I}(x_1,\dots,x_n,t),\cdot\cdot\right)\end{aligned}$$

be the result of substitution. Then

$$\frac{\partial\mathcal{F}}{\partial x_i}=\frac{\mathrm{d}}{\mathrm{d}x_i}F=F_{x_i}+\sum w_{Ii}^jF_{w_I^j}+\sum w_{Iit}^jF_{w_{It}^j}+\cdots \qquad (i=1,\dots,n),$$

$$\frac{\partial\mathcal{F}}{\partial t}=\frac{\mathrm{d}}{\mathrm{d}t}F=F_t+\sum w_{It}^jF_{w_I^j}+\sum w_{Itt}^jF_{w_{It}^j}+\cdots$$

in terms of *total derivatives* $d/\mathrm{d}x_i, d/\mathrm{d}t$. The iterations

$$\frac{\mathrm{d}}{\mathrm{d}x_I}=\frac{\mathrm{d}}{\mathrm{d}x_{i_1}}\cdots\frac{\mathrm{d}}{\mathrm{d}x_{i_r}},\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}x_I}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}x_{i_1}}\cdots\frac{\mathrm{d}}{\mathrm{d}x_{i_r}} \quad (I=i_1\cdots i_r)$$

make a good sense, too. In accordance with the common practice, the presence of various substitutions need not be always explicitly declared since it will be clear from the context.

With this preparation, let $f=f(\cdot\cdot,x_i,w_I^j,\cdot\cdot)$ be a fixed function of variables (1), the *Lagrange function.* Repeated use of the rule

$$g\,w_{Iit}^j=-\frac{\mathrm{d}}{\mathrm{d}x_i}g\cdot w_{It}^j+\frac{\mathrm{d}}{\mathrm{d}x_i}(g\,w_{It}^j)$$

yields the *variational identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}f = \sum f_{w_I^j} w_{It}^j = e[f] + d[f] \quad (e[f] = \sum e^j[f] w_t^j, d[f] = \sum \frac{\mathrm{d}}{\mathrm{d}x_i}F_i) \tag{2}$$

where

$$e^j[f] = \sum (-1)^r \frac{\mathrm{d}}{\mathrm{d}x_I} f_{w_I^j} \quad (j = 1, \ldots, m;\ I = i_1 \cdots i_r;\ i_1 \le \ldots \le i_r) \tag{3}$$

are the *Euler–Lagrange expressions* and $d[f]$ the *divergence component* with (not uniquely determined) coefficients F_i linearly depending on variations w_{It}^j.

Proposition 1.1 (uniqueness)**.** *Let*

$$\frac{\mathrm{d}}{\mathrm{d}t}f = \sum e^j w_t^j + \sum \frac{\mathrm{d}}{\mathrm{d}x_i}G_i \tag{4}$$

where e^j are functions of variables (1) *while G_i may also depend on variations. Then $e^j[f] = e^j$; $j = 1, \ldots, m$; and $d[f] = \sum \frac{\mathrm{d}}{\mathrm{d}x_i}G_i$.*

Proof. Assuming (4), substituting $w^j = w^j(x_1, \ldots, x_n, t)$ and denoting

$$\mathrm{d}x = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n,\ \mathrm{d}x^i = -(-1)^i \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n,$$

we obtain identity

$$\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} f \mathrm{d}x = \int_\Omega (e[f] + d[f]) \mathrm{d}x = \sum \int_\Omega e^j w_t^j \mathrm{d}x + \sum \int_{\partial\Omega} G_i \mathrm{d}x^i$$

by integration over a domain Ω in the space of independent variables. Therefore

$$\sum \int_\Omega (e^j - e^j[f]) w_t^j \mathrm{d}x + \sum \int_{\partial\Omega} (G_i - F_i) \mathrm{d}x^i = 0$$

identically, for all variations. The following classical argument may be applied: if variations $w_t^j = w_t^j(x_1, \ldots, x_n, t)$ are vanishing near boundary $\partial\Omega$, the second summand disappears and this implies the desired result. □

Proposition 1.2 (divergence)**.** *We claim that $e[f] = 0$ if and only if*

$$f = \sum \frac{\mathrm{d}}{\mathrm{d}x_i} f_i$$

where $f_1, \ldots, f_n$ are appropriate functions of variables (1).

Proof. Assuming $e[f] = 0$, let us insert the expressions

$$tw^j + (1 - t)c^j \quad (j = 1, \ldots, m;\ c^j = c^j(x_1, \ldots, x_n)) \tag{5}$$

(c^j may be arbitrary but fixed functions) for variables w^j into identity (2). We obtain

$$f|_{t=1} - f|_{t=0} = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f \,\mathrm{d}t = \int_0^1 \sum \frac{\mathrm{d}}{\mathrm{d}x_i} F_i \,\mathrm{d}t = \sum \frac{\mathrm{d}}{\mathrm{d}x_i} \int_0^1 F_i \,\mathrm{d}t \tag{6}$$

by subsequent integration where the first term ($t = 1$) is identical with the original function f, the second term ($t = 0$) is a certain divergence

$$c = c(x_1, \ldots, x_n) = \frac{\mathrm{d}}{\mathrm{d}x_1} \int c \,\mathrm{d}x_1$$

and the right–hand integrals are functions of variables (1).

The converse is easier since trivially

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \frac{\mathrm{d}}{\mathrm{d}t} \sum \frac{\mathrm{d}}{\mathrm{d}x_i} f_i = \sum \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{\mathrm{d}}{\mathrm{d}t} f_i$$

and the uniqueness implies $e[f] = 0$. □

Proposition 1.3 (Helmholz)**.** *Let* $e^1, \ldots, e^m$ *be functions of variables* (1)*. Identity*

$$e[F] = 0 \quad (F = \sum e^j w^j_t) \tag{7}$$

in the extended jet space is satisfied if and only if

$$e^j = e^j[f] \quad (j = 1, \ldots, m) \tag{8}$$

for an appropriate Lagrange function f *of variables* (1)*.*

Before passing to the proof, let us discuss identity (7). In the *extended jet space*, we have Euler–Lagrange expressions

$$e^j[F] = \sum (-1)^r \frac{\mathrm{d}}{\mathrm{d}x_K} F_{w^j_K} \qquad (j = 1, \ldots, m;\ K = k_1 \cdots k_r;\ r = 0, 1, \ldots)$$

where $k_1, \ldots, k_r = 1, \ldots, n+1$. We prefer notation x_{n+1} for the parameter t here. Clearly

$$e^{j'}[F] = \sum F^{jj'}_I w^j_{It}$$

where $F^{jj'}_I$ are functions only of variables (1). It follows that identity (7) is equivalent to the *Helmholz condition*

$$F^{jj'}_I = 0 \quad (\text{all } j, j' \text{ and } I) \tag{9}$$

for the given functions $e^1, \ldots, e^m$. It should be noted that there are many formally rather dissimilar and cumbersome transcriptions of the Helmholz condition in actual literature. Our new record (7) is unusually simple and the shortest one.

Proof. Identity (2) reads

$$F = \frac{\mathrm{d}}{\mathrm{d}t} f - \sum \frac{\mathrm{d}}{\mathrm{d}x_i} F_i \quad (F = e[f] = \sum e^j[f] w^j_t),$$

hence $e[F] = 0$ if F is regarded as a Lagrange function in the extended jet space and Proposition 1.1 is applied.

Let us conversely assume (7), therefore

$$F = \sum \frac{\mathrm{d}}{\mathrm{d}x_i} G_i + \frac{\mathrm{d}}{\mathrm{d}t} G \, (= \sum \frac{\mathrm{d}}{\mathrm{d}x_i} G_i + \frac{\mathrm{d}}{\mathrm{d}x_{n+1}} G) \tag{10}$$

for appropriate functions G_i and G if Proposition 1.2 is applied in the extended jet space. Applying the uniqueness, we have $F = e[f]$ where $f = G$ and our task is to prove that this vague G is a function of variables (1).

For this aim, let us write x_{n+1} for the additional variable in the extended jet space while t will denote the new variation as follows:

$$F = \sum e^{j'} w^{j'}_{n+1},\ \frac{dF}{\mathrm{d}t} = \sum \frac{\partial e^{j'}}{\partial w^j_I} w^j_{It} w^{j'}_{n+1} + \sum e^{j'} w^{j'}_{n+1,t}.$$

We suppose $e[F] = 0$ whence

$$\frac{dF}{\mathrm{d}t} = \sum \frac{\mathrm{d}}{\mathrm{d}x_i} F_i + \frac{\mathrm{d}}{\mathrm{d}x_{n+1}} \sum e^j w^j_t \qquad (\text{certain } F_i)$$

is a mere divergence. Analogously as in (6), we obtain

$$F = F|_{t=1} = \{F|_{t=0} + \sum \frac{\mathrm{d}}{\mathrm{d}x_i} \int_0^1 F_i \mathrm{d}t\} + \frac{dG}{\mathrm{d}x_{n+1}} \qquad (G = \sum \int_0^1 e^j \mathrm{d}t \, (w^j - c^j))$$

identical with (10) since the summand $\{\cdots\}$ is a divergence. However this function G indeed depends only on variables (1) □

Assuming Proposition 1.3 for proved, much shorter way to the same function G is as follows.

Proposition 1.4 (Tonti)**.** *Let us introduce the Lagrange function*

$$\tilde{f} = \int_0^1 e[f]\,\mathrm{d}t = \sum \int_0^1 e^j[f]\,\mathrm{d}t\,(w^j - c^j) \tag{11}$$

where variables (5) *were inserted into the integral. Then* $e[\tilde{f}] = e[f]$.

Proof. Equation (2) together with substitution (5) and integration implies

$$f|_{t=1} - f|_{t=0} = \int_0^1 e[f]\,\mathrm{d}t + \sum \frac{\mathrm{d}}{\mathrm{d}x_i}\int_0^1 F_i\,\mathrm{d}t = \tilde{f} + \sum \frac{\mathrm{d}}{\mathrm{d}x_i}\int_0^1 F_i\,\mathrm{d}t.$$

The first summand ($t = 1$) is identical with the original function f, the second one ($t = 0$) is merely a function $c = c(x_1, \ldots, x_n)$ of independent variables. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{f} + \frac{\mathrm{d}}{\mathrm{d}t}\sum \frac{\mathrm{d}}{\mathrm{d}x_i} f_i = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{f} + \sum \frac{\mathrm{d}}{\mathrm{d}x_i}\frac{df_i}{\mathrm{d}t} \quad (f_i = \int_0^1 F_i\,\mathrm{d}t)$$

and the proof is done by applying Propositions 1.2 and 1.1. □

We have in principle resolved the *exact inverse problem*: if $e^1 = \cdots = e^n = 0$ is a given system of differential equations, the *Helmholz conditions* (9) are necessary and sufficient for the existence of a (uncertain here) Lagrange function f satisfying (8). Then the *Tonti integral* (11) provides the explicit solution of the problem since $e[\tilde{f}] = e[f]$. This is a very forceful solution, alas, not the best one owing to the fatal inequality

$$\text{order } \tilde{f} \leq \text{ order } e^j[f] \leq 2 \text{ order } f$$

where the important relationship between the Tonti solution $\tilde{f}$ and the original function f is passed over in full silence.

For better clarity, let us mention the "introductory" Lagrange function $f = f(x, y, y')$ of the classical calculus of variations where the jet coordinates are replaced by the common notation at this point. The Euler–Lagrange expression

$$e^1[f] = f_y - (f_{y'})' = f_y - f_{y'x} - f_{y'y}y' - f_{y'y'}y'' = E(x, y, y', y'')$$

provides the Tonti integral

$$\tilde{f} = \int_0^1 E\left(x, ty + (1-t)c, ty' + (1-t)c', ty'' + (1-t)c''\right)\mathrm{d}t\,(y - c)$$

where $c = c(x)$ is a fixed (in principle arbitrary) function. If function E is given in advance, then $\tilde{f}$ as a rule essentially differs from the primary function f. For instance, if we choose elementary

$$f = \frac{e^y}{y'},\; e^1[f] = 2\frac{e^y}{y'}\left(1 - \frac{y''}{(y')^2}\right)$$

then the Tonti integral

$$\tilde{f} = 2\int_0^1 \frac{e^{ty+(1-t)x}}{ty' + (1-t)}\left(1 - \frac{ty''}{(ty' + 1 - t)^2}\right)\mathrm{d}t\,(y - x)$$

$$= 2\int_1^{y'} e^{(\tau-1)\frac{y-x}{y'-1} + x}\left(1 - \frac{\tau - 1}{\tau^2}\frac{y''}{y' - 1}\right)\frac{d\tau}{\tau}\,\frac{y - x}{y' - 1}$$

(where $c = c(x) = x$) is a terrible higher transcendence. Though this result can be at least in principle improved by certain Tonti–like corrections [12], the sought function f remain lost in very complicated formulae. We discuss other approach later on: if the above function E is given, we know the second derivative $f_{y'y'}$ and f can be found by quadrature and y'–linear correction. One can then easily obtain even the *global result*

$$f(x,y,y') = \frac{e^y}{y'} + \frac{\mathrm{d}}{\mathrm{d}t}F(x,y) \qquad (y' \neq 0)$$

where F may be arbitrary function. In fact we have *two functions* F : one for the domain $y' > 0$ and other if $y' < 0$. In the theory to follow, we obtain even the "ramified"global solutions by using de Rham theory.

On the other hand, in spite of the fact that the Propositions are insufficient in some respects, the easy Proofs deserve more attention. Indeed, the Helmholz conditions were already proved by many methods [6],[12],[13],[10],[14]: the potential operators, the variational bicomplex, the Helmholz–Sonin mapping, analysis of Poincaré–Cartan form, and so on. It is however only our naive approach that can be still developed as follows.

Let us complete the identity (2) with the intermediate terms,

$$\frac{\mathrm{d}}{\mathrm{d}t}f = \sum f_{w_I^j} w_{It}^j = \cdots = \sum e_{I_j}^j[f] w_{I_j t}^j + \mathcal{D} = \cdots = e[f] + d[f], \tag{12}$$

appearing in the calculation.The sum in the middle term runs over $j = 1, \ldots, m$ and all multiindices I_j belong to a certain set $\mathcal{I}(j)$. In particular

$$e_{I_j}^j[f] = f_{w_I^j},\ I_j \in \mathcal{I}(j) = \{I : |I| \leq \operatorname{order} f\},\ \mathcal{D} = 0,$$

for the initial term of (12) and

$$e_\phi^j[f] = e^j[f],\ \phi = I_j \in \mathcal{I}(j) = \{\phi\},\ \mathcal{D} = d[f],$$

for the last term. The vague notation $\mathcal{D}$ of all divergences is sufficient here. (There are *many* possible sequences (12) corresponding to various strategies of the calculation. We consider only one of them here.)

Theorem 1.5. *For any intermediate term, the identity*

$$e[\mathcal{F}[f]] = 0 \qquad (\mathcal{F}[f] = \sum e_{I_j}^j[f] w_{I_j t}^j, j = 1, \ldots, m; I_j \in \mathcal{I}(j)) \tag{13}$$

in the extendend jet space holds true.

Theorem 1.6. *Let certain functions* $e_{I_j}^j$ $(j = 1, \ldots, m; I_j \in \mathcal{I}(j))$ *of variables* (1) *satisfy the generalized Helmholz condition*

$$e[\mathcal{F}] = 0 \qquad (\mathcal{F} = \sum e_{I_j}^j w_{I_j t}^j, j = 1, \ldots, m; I_j \in \mathcal{I}(j)) \tag{14}$$

in the extended jet space. Then $\mathcal{F} = \mathcal{F}[\bar{f}] + \mathcal{D}$ *where*

$$\bar{f} = \sum \int_0^1 e_{I_j}^j \mathrm{d}t \left(w_{I_j}^j - \frac{\partial}{\partial x_{I_j}} c^j \right) \qquad (j = 1, \ldots, m; I_j \in \mathcal{I}(j)) \tag{15}$$

with variables (5) *inserted into the integral.*

Theorem 1.7. *For any fixed* $j = 1, \ldots, m$ *and all intermediate terms with* $\mathcal{I}(j) = \{\phi\}$, *the coefficients*

$$e_{I_j}^j[f] = e_\phi^j[f] = e^j[f]$$

are equal.

All proofs are literally almost the same as above. The next remark should be nevertheless useful [11]. In the common inverse problem, all Euler–Lagrange expressions are given. However assume that (e.g.) only the first function $e^1 = e^1[f]$ is prescribed and let $m \geq 2$. Then we may put

$$\mathcal{F} = e^1 w_t^1 + \sum f_{w_{I_k}^k} w_{I_k t}^k \qquad (k = 2, \ldots, m; |I_k| \leq \operatorname{order} f)$$

and *the Helmholz condition* (14) *with the Tonti integral* (15) *resolve such "incomplete" inverse problem.*

2 The Global Strategy

We leave the Helmholz solvability condition and the Tonti formula from now on. We are interested even in the *global solvability* and *direct calculation* of the "economical" solution without any correction. Then the exact inverse problem cannot be regarded as trivially as it seemed at a first glance. Let us therefore focus only on the modest task here, namely on the problem of whether or not a given *second–order* system of differential equations is *identical* with the Euler–Lagrange equations of a *first–order* variational integral.

The jet variables (1) are enough from now on. We are going to deal systematically with the first–order Lagrange function $f = f(\cdot\cdot, x_i, w^j, w^j_i, \cdot\cdot)$. Then the Euler–Lagrange expressions are

$$e^j[f] = E^j[f] - \sum E^{jj'}_{ii'}[f] w^{j'}_{ii'} \quad (j = 1, \ldots, m)$$

where

$$E^j[f] = f_{w^j} - \sum f_{w^j_i x_i} - \sum f_{w^j_i w^{j'}} w^{j'}_i, \; E^{jj'}_{ii'}[f] = \frac{1}{2}\left(f_{w^j_i w^{j'}_{i'}} + f_{w^j_{i'} w^{j'}_i}\right).$$

The exact inverse problem is as follows. Let

$$F^{jj'}_{ii'}(\cdot), F^j(\cdot) \qquad (i, i' = 1, \ldots, n; \; j, j' = 1, \ldots, m; \; (\cdot) = (\cdot\cdot, x_i, w^j, w^j_i, \cdot\cdot))$$

be given functions. We ask the question whether the requirements

$$F^{jj'}_{ii'} = E^{jj'}_{ii'}[f], \; F^j = E^j[f] \quad (i, i' = 1, \ldots, n; \; j, j' = 1, \ldots, m) \tag{16}$$

are satisfied for an appropriate first–order Lagrange function f.

In order to simplify the notation, we abbreviate

$$f^j_i = f_{w^j_i}, \; f^{jj'}_{ii'} = f_{w^j_i w^{j'}_{i'}}, \; \ldots, \; f^j_{ix_{i'}} = f'_{w^j_i x_{i'}}, \; \ldots$$

from now on. Let us moreover introduce the auxiliary functions

$$G^{jj'}_{ii'} = \frac{1}{2}\left(f^{jj'}_{ii'} - f^{jj'}_{i'i}\right) = \frac{1}{2}\left(f_{w^j_i w^{j'}_{i'}} - f_{w^j_{i'} w^{j'}_i}\right).$$

Then

$$f^{jj'}_{ii'} = F^{jj'}_{ii'} + G^{jj'}_{ii'} \quad (i, i' = 1, \ldots, n; \; j, j' = 1, \ldots, m).$$

The symmetry properties

$$F^{jj'}_{ii'} = F^{jj'}_{i'i} = F^{j'j}_{i'i} = F^{j'j}_{ii'}, \; G^{jj'}_{ii'} = -G^{jj'}_{i'i} = G^{j'j}_{i'i} = -G^{j'j}_{ii'} \tag{17}$$

are postulated.

We pass to the topic proper.

The *first requirement* (16) is equivalent to the Pfaffian equations

$$d_1 f = \sum f^j_i \mathrm{d}w^j_i, \; d_1 f^j_i = \sum f^{jj'}_{ii'} \mathrm{d}w^{j'}_{i'} \quad (f^{jj'}_{ii'} = F^{jj'}_{ii'} + G^{jj'}_{ii'}) \tag{18}$$

where differential d_1 is applied only to the first–order variables. Due to the symmetry properties (17), the first Pfaffian equation is always solvable. The second system of the Pfaffian equations is locally solvable if and only if

$$f^{jj'j''}_{ii'i''} = \left(f^{jj'}_{ii'}\right)_{w^{j''}_{i''}} = \left(f^{jj''}_{ii''}\right)_{w^{j'}_{i'}} = f^{jj''j'}_{ii''i'}$$

hence

$$(F^{jj'}_{ii'} + G^{jj'}_{ii'})_{w^{j''}_{i''}} = (F^{jj''}_{ii''} + G^{jj''}_{ii''})_{w^{j'}_{i'}} \quad (i, i' = 1, \ldots, n; \; j = 1, \ldots, m), \tag{19}$$

where $F^{jj'}_{ii'}$ are given but $G^{jj'}_{ii'}$ unknown functions. The first requirement (16) is regarded as clarified at this point.

Turning to the *second requirement* (16), it implies

$$(F^j)_{w^{j'}_{i'}} = f^{j'}_{i'w^j} - \sum f^{jj'}_{ii'x_i} - \sum f^{jj'}_{ii'w^{j''}} w^{j''}_i - f^j_{i'w^{j'}}. \tag{20}$$

Denoting

$$F^{j'j}_{i'} = (F^j)_{w^{j'}_{i'}} + \sum f^{jj'}_{ii'x_i} + \sum f^{jj'}_{ii'w^{j''}} w^{j''}_i \ \ (= f^{j'}_{i'w^j} - f^j_{i'w^{j'}}), \tag{21}$$

there are obvious identities

$$F^{j'j}_{i'} + F^{jj'}_{i'} = 0,\ (F^{j'j}_{i'})_{w^{j''}_{i''}} = f^{j'j''}_{i'i''w^j} - f^{jj''}_{i'i''w^{j'}}. \tag{22}$$

Identities (22) can be expressed in terms of functions

$$F^j,\ F^{jj'}_{ii'},\ G^{jj'}_{ii'} \quad (i, i' = 1, \ldots, n;\ j, j' = 1, \ldots, m) \tag{23}$$

and therefore may be regarded as necessary solvability conditions for the second requirement (16). Alas, they are not sufficient.

Indeed, assume the second identity (22). Then

$$F^{j'j}_{i'} = f^{j'}_{i'w^j} - f^j_{i'w^{j'}} + B^{j'j}_{i'} \tag{24}$$

where

$$B^{j'j}_{i'} = B^{j'j}_{i'}(\cdot\cdot, x_i, w^{j''}, \cdot\cdot) \quad (i' = 1, \ldots, n;\ j, j' = 1, \ldots, m)$$

are appropriate functions. With this result, the definition equation (21) reads

$$(F^j)_{w^{j'}_{i'}} = F^{j'j}_{i'} - \sum f^{jj'}_{ii'x_i} - \sum f^{jj'}_{ii'w^{j''}} w^{j''}_i = \frac{\partial}{\partial w^{j'}_{i'}} E^j[f] + B^{j'j}_{i'}$$

(direct verification) and therefore

$$F^j = E^j[f] + \sum B^{j'j}_{i'}(w^{j'}_{i'} - c^{j'}_{i'}) + B^j \tag{25}$$

where

$$B^j = B^j(\cdot\cdot, x_i, w^{j'}, \cdot\cdot) \quad (j = 1, \ldots, m)$$

are appropriate functions. The functions

$$c^j_i = c^j_i(\cdot\cdot, x_{i'}, w^{j'}, \cdot\cdot) \qquad (i = 1, \ldots, n;\ j = 1, \ldots, m) \tag{26}$$

are fixed and may be arbitrarily chosen in advance. The second requirement (16) is satisfied if and only if $B^{j'j}_{i'} = B^j = 0$ identically and this goal can be achieved as follows.

Identity (17) may be regarded as a system of differential equations

$$\frac{\partial^2 f}{\partial w^j_i \partial w^{j'}_{i'}} = F^{jj'}_{ii'} + G^{jj'}_{ii'} \quad (i, i' = 1, \ldots, n;\ j, j' = 1, \ldots, m\,). \tag{27}$$

If $\bar{f} = \bar{f}(\cdot\cdot, x_i, w^j, w^j_i, \cdot\cdot)$ is the (unique) particular solution such that

$$\bar{f} = \frac{\partial \bar{f}}{\partial w^j_i} = 0\ (i = 1, \ldots, n;\ j = 1, \ldots, m) \quad \text{if} \quad w^1_1 = c^1_1, \ldots, w^m_n = c^m_n \tag{28}$$

then the general solution is

$$f = \bar{f} + \sum b^j_i (w^j_i - c^j_i) + b \tag{29}$$

where

$$b^j_i = b^j_i(\cdot\cdot, x_{i'}, w^{j'}, \cdot\cdot), b = b(\cdot\cdot, x_{i'}, w^{j'}, \cdot\cdot) \quad (i = 1, \ldots, n;\ j = 1, \ldots, m)$$

are arbitrary functions. One can then see that identity (24) with $B^{jj'}_{i'}=0$ is equivalent to the equation

$$F^{j'j}_{i'}=(b^{j'}_{i'})_{w^j}-(b^{j}_{i'})_{w^{j'}}\quad\text{if}\quad w^1_1=c^1_1,\dots,w^m_n=c^m_n$$

for the coefficients b^j_i. With the latent use of the first identity (22), this is expressed by the *global* equation

$$\sum F^{j'j}_{i'}\mathrm{d}w^j\wedge\mathrm{d}w^{j'}=2d_0\sum b^{j'}_{i'}\mathrm{d}w^{j'}\quad\text{if}\quad w^1_1=c^1_1,\dots,w^m_n=c^m_n \tag{30}$$

where differential d_0 is applied to variables $w^1,\dots,w^m$. Due to the Poincaré Lemma, we have necessary and sufficient condition

$$d_0\sum F^{j'j}_{i'}\mathrm{d}w^j\wedge\mathrm{d}w^{j'}=0\ (i'=1,\dots,n)\quad\text{if}\quad w^1_1=c^1_1,\dots,w^m_n=c^m_n \tag{31}$$

for the *local* existence of functions b^j_i. Quite analogously, assuming already $B^{j'j}_{i'}=0$, equation (25) with $B^j=0$ is ensured if and only if

$$F^j=b_{w^j}-\sum(b^j_i)_{x_i}-\sum(b^j_i)_{w^{j''}}w^{j''}_i\quad\text{if}\quad w^1_1=c^1_1,\dots,w^m_n=c^m_n.$$

This is equivalent to the *global* identity

$$\sum\left\{F^j+\sum(b^j_i)_{x_i}+\sum(b^j_i)_{w^{j''}}c^{j''}_i\right\}\mathrm{d}w^j=d_0b \tag{32}$$

and we have necessary and sufficient condition

$$d_0\sum\{\cdots\}\,\mathrm{d}w^j=0\quad\text{if}\quad w^1_1=c^1_1,\dots,w^m_n=c^m_n \tag{33}$$

for the *local* existence of function b.

If (in principle arbitrary) functions (26) depend only on variables $x_1,\dots,x_n$, condition (33) can be expressed without the use of coefficients b^j_i. Indeed, we may substitute

$$d_0\sum(b^j_i)_{x_i}\mathrm{d}w^j=\frac{\partial}{\partial x_i}d_0\sum b^j_i\mathrm{d}w^j=\frac{1}{2}\frac{\partial}{\partial x_i}\sum F^{j'j}_i\mathrm{d}w^j\wedge\mathrm{d}w^{j'},$$

$$d_0\sum(b^j_i)_{w^{j''}}c^{j''}_i\mathrm{d}w^j=c^{j''}_i\frac{\partial}{\partial w^{j''}}d_0\sum b^j_i\mathrm{d}w^j=\frac{1}{2}c^{j''}_i\frac{\partial}{\partial w^j}\sum F^{j'j}_i\mathrm{d}w^j\wedge\mathrm{d}w^{j'}$$

into (33) to obtain the concluding condition

$$2d_0\sum F^j\mathrm{d}w^j+\sum\left(\frac{\partial}{\partial x_i}+\sum c^{j''}_i\frac{\partial}{\partial w^{j''}}\right)F^{j'j}_i\mathrm{d}w^j\wedge\mathrm{d}w^{j'}=0 \tag{34}$$

which is equivalent to (33).

The main idea of our strategy is done. At a first glance, our achievements look as follows.

Theorem 2.1 (local)**.** *The first–order Lagrange function f resolving the exact inverse problem* (16) *is given by* (29) *where $\bar{f}$ is a particular solution of equation* (27) *satisfying the initial conditions* (28) *and the coefficients b^j_i, b satisfy* (30) *and* (32)*. The necessary and sufficient solvability conditions for the local existence of the solution f are* (19), (22), (31) *and* (33) *or* (34)*. They are expressed in terms of functions* (23) *where F^j and $F^{jj'}_{ii'}$ are given but $G^{jj'}_{ii'}$ are unknown.*

The existence of a *special local solution* of the inverse problem (16) was already obtained [6, Theorem 3.2] by the traditional Helmholz condition and rather ingenious and toilsome correction of the Tonti formula. Our approach is of other nature. We are interested in *all local* and even in *all global solutions* which compels the corresponding deep reconstruction of the common methods. The succinct but rather dark Helmholz condition is omitted. We instead introduce several more explicit solvability requirements adapted to the global theory. The explicit but in fact misleading Tonti formula is replaced with three classical tasks to be still investigated: the system (27) with the primary unknown function f and the new auxiliary functions $G_{ii'}^{jj'}$, moreover the global identities (30), (32) with unknown functions b_i^j and b. Then the deeper insight at the result is as follows.

Summary (global). It consists of points (ι)–($\nu\iota\iota\iota$) for better references.

(ι) *The underlying domain.* The calculations are performed on an open subset $\mathcal{A} \subset J(P)$ of the first–order fibered jet space $\pi_1 : J(P) \to M$ of the fibered manifold $\pi_0 : P \to M$ where

$$J(P) = \mathbb{R}^{m+n+mn}, \text{ coordinates } x_i, w^j, w_i^j;$$
$$P = \mathbb{R}^{m+n}, \text{ coordinates } x_i, w^j; \; M = \mathbb{R}^n, \text{ coordinates } x_i$$

($i = 1, \ldots, n; j = 1, \ldots, m; m, n \geq 1$). The projections $\pi_1 : J(P) \to M$ and $\pi_0^1 : J(P) \to P$ can be naturally restricted to the subset $\mathcal{A} \subset J(P)$.

($\iota\iota$) *The cross–section.* We postulate the existence of a cross–section $\mathcal{B} \subset \mathcal{A}$ of the π_0^1–fibration of $\mathcal{A}$ schematically illustrated in Figure 1a. It is given by equations

$$w_i^j = c_i^j(\cdot\cdot, x_{i'}, w^{j'}, \cdot\cdot) \qquad (i, i' = 1, \ldots, n; j, j' = 1, \ldots, m). \tag{35}$$

Alternatively saying, the π_0^1–fibers denoted $\mathcal{F}(B)$ of $\mathcal{A}$ are parametrized by the points $B = \mathcal{F}(B) \cap \mathcal{B}$.

($\iota\iota\iota$) *Still a cross–section.* The projection $\pi_0 : P \to M$ induces certain π_0–fibration of $\mathcal{B}$ (use the coordinates $x_{i'}, w^{j'}$ in (35) on $\mathcal{B}$) and we suppose the existence of cross–section $\mathcal{C}$ of $\mathcal{B}$ given by certain equations

$$w^{j'} = c^{j'}(\cdot\cdot, x_{i'}, \cdot\cdot) \qquad (i' = 1, \ldots, n; j' = 1, \ldots, m). \tag{36}$$

Alternatively saying, the π_0–fibers $\mathcal{F}(C)$ of $\mathcal{B}$ are parametrized by the points $C = \mathcal{F}(C) \cap \mathcal{C}$, see Figure 1b.

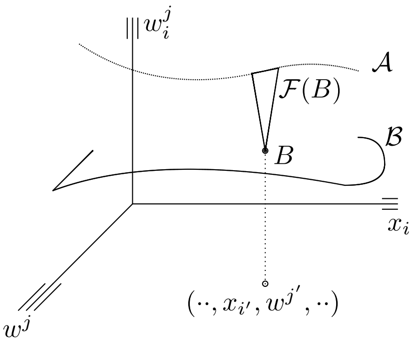

Figure 1a

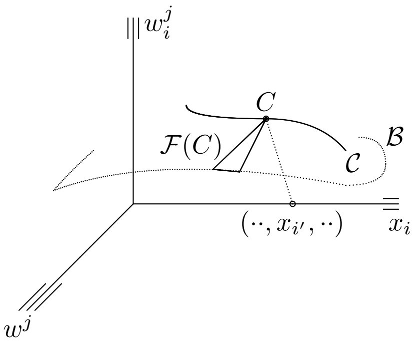

Figure 1b

($\iota\nu$) *The resolving system.* Equations (27)–(29) are equivalent to the system

$$\frac{\partial^2 f}{\partial w_i^j \partial w_{i'}^{j'}} = F_{ii'}^{jj'} + G_{ii'}^{jj'} \qquad (i, i' = 1, \ldots, n; j = 1, \ldots, m) \tag{37}$$

in $\mathcal{A}$ with initial conditions

$$f(B) = b(B), \frac{\partial f}{\partial w_i^j}(B) = b_i^j(B) \qquad (B \in \mathcal{B}; i = 1, \ldots, n; j = 1, \ldots, m) \tag{38}$$

at the cross–section $\mathcal{B}$. While $F_{ii'}^{jj'}$ are given functions, $G_{ii'}^{jj'}$ should be still determined. This is a delicate point, however, it is sufficient to employ only *one particular choice* of these functions $G_{ii'}^{jj'}$ in order to obtain *all solutions* of the inverse problem. Alternatively saying, *it is sufficient to determine only one particular choice of the functions* $G_{ii'}^{jj'}$ *such that the compatibility conditions* (19) *hold true.*

(ν) *The global solution.* Assuming the fibers $\mathcal{F}(B)$ of $\mathcal{A}$ connected, the sought function f with the given second–order derivatives (37) and the initial conditions (38) is given by a familiar quadrature (not written here) on every fiber $\mathcal{F}(B)$. It follows that the homologies $H_1(\mathcal{F}(B))$ completely describe the global situation, the "ramification" of f. In particular the vanishing $H_1(\mathcal{F}(B)) = 0$ ensures the single–valued solution.

($\nu\iota$) *The global initial values.* Quite analogously, equations (30) and (32) determine the functions b_i^j and b on every fiber $\mathcal{F}(C)$. The solution is globally described by the cohomologies $H^2(\mathcal{F}(C))$ and $H^1(\mathcal{F}(C))$, respectively.This is a matter of the familiar de Rham theory [15],[16].

($\nu\iota\iota$) *On the de Rham theory.* Our homologies over the field $\mathbb{R}$ are dual to the cohomologies and the duality pairing is expressed here by certain explicit integrals over the cycles representing the homologies. However, they depend on the parameters $B \in \mathcal{B}$ and $C \in \mathcal{C}$. In other words, we deal with the "fibered" de Rham theory, the cohomologies of fibers of a certain underlying fibered space. This is a remarkable novelty which was not yet systematically investigated. It resembles to some extent the familiar characteristic classes of vector bundles.

($\nu\iota\iota\iota$) *The singular points.* The isolated singular points of functions $F_{ii'}^{jj'}$ affect the homologies $H_1(\mathcal{F}(C))$ only if $\dim \mathcal{F}(C) = mn \leq 2$. Analogously singular points of functions F^j cause some difficulties only if $\dim \mathcal{F}(B) = m \leq 3$. Otherwise no ramification of the solution near the singular points appears.

The summary is done and we intentionally do not try to represent our task by a weighty Theorem. The above results should be regarded as a declaration of perspectives since several new aspects of the exact inverse problem appears here for the first time and deserve much thorough and systematical study. The following remarks should be useful in this respect.

Calculation on the fibers $\mathcal{F}(B)$. Equations (37) are equivalent to the Pfaffian system (18) Assuming the solvability (19), the *global* Frobenius theorem can be applied and the function f can be obtained by integration in accordance with the classical mathematical analysis on every fiber $\mathcal{F}(B)$. The postulated cross–section $\mathcal{B}$ and the connectivity of fibers ensure the smooth dependence of the result f on the parameter $B \in \mathcal{B}$. The function f may be in general multiply–valued. The only difficult problem remains, the determination of unknown functions $G_{ii'}^{jj'}$ and we refer to examples below.

Calculation on the fibers $\mathcal{F}(C)$. The solutions of equation (30) and (31) with unknown functions b_i^j and b are a matter of classical de Rham theory. For instance, the integrals

$$\iint \sum F_{i'}^{j'j} \mathrm{d}w^j \wedge \mathrm{d}w^{j'} \qquad (d_0 \sum F_{i'}^{j'j} \mathrm{d}w^j \wedge \mathrm{d}w^{j'} = 0)$$

over every two–dimensional cycle in $\mathcal{F}(\mathcal{C})$ are vanishing *if and only if* there exists global univalent solution b_i^j. In other case we obtain precise information of multivalent solutions in terms of de Rham cohomology $H^2(\mathcal{F}(\mathcal{C}))$. So we have necessary and sufficient conditions together with the overview of all solutions.

On the global cross–sections. We postulate the global cross–sections $\mathcal{B}$ and $\mathcal{C}$ in order to ensure the smooth dependence on parameters $B \in \mathcal{B}$ and $C \in \mathcal{C}$. In other case, much deeper and not yet available de Rham theory depending on parameters is necessary.

The isolated singularities of various orders may be extracted from the underlying space and do not cause much difficulties. Since we deal only with homologies H^1 and H^2, they do not affect the results for higher values n and m.

The above achievements look involved rather than of any practical use. Let us nevertheless turn to particular subcases in order to obtain convincing results and to gain more experience. We formally focus on the local aspects, however, the global aspects are in fact latently present as well. Some simple examples shall be also compiled though they are not common in actual literature.

3 The Subcase of One Dependent Variable

We suppose $m = 1$. Let us abbreviate

$$w_I = w_I^1, f_i = f_i^1 = f_{w_i^1}, f_{ii'} = f_{ii'}^{11} = f_{w_i^1 w_{i'}^1}, \ldots$$

but the remaining notation is retained. Then

$$f = f(\cdot\cdot, x_i, w, w_i, \cdot\cdot),\ e[f] = e^1[f]w_t,\ e^1[f] = E^1[f] - \sum E^{11}_{ii'}[f]w_{ii'}$$

where

$$E^{11}_{ii'} = f_{ii'},\ E^1[f] = f_w - \sum f_{ix_i} - \sum f_{iw}w_i.$$

The exact inverse problem

$$F^{11}_{ii'} = E^{11}_{ii'}[f] \quad (i, i' = 1, \dots, n),\ F^1 = E^1[f] \tag{39}$$

simplifies since $G^{11}_{ii'} = 0$. The *first requirement* (39) is solvable if and only if

$$(F^{11}_{ii'})_{w_{i''}} = (F^{11}_{ii''})_{w_{i'}} \qquad (i, i', i'' = 1, \dots, n) \tag{40}$$

where the symmetry $F^{11}_{ii'} = F^{11}_{i'i}$ is supposed. Passing to the *second requirement* (39), one can infer that the vanishing $F^{11}_{i'} = 0$ follows from the first identity (22) and then the definition (21) turns into the single solvability condition

$$(F^1)_{w_{i'}} + \sum (F^{11}_{ii'})_{x_i} + \sum (F^{11}_{ii'})_w w_i = 0 \qquad (i' = 1, \dots, n). \tag{41}$$

Condition (41) can be alternatively regarded as a (compatible) system of differential equations for the function F^1 in terms of given functions $F^{11}_{ii'}$ satisfying (40). The second identity (22) becomes trivial since $j = j' = j'' = 1$. Let us pass to the existence of coefficients b^1_i and b. First of all, (30) is simplified to $d_0 \sum b^1_i \mathrm{d}w = 0$ whence $b^1_i = b^1_i(x_1, \dots, x_n, w)$ may be arbitrary functions. Therefore only one additional equation

$$b_w = F^1 + \sum (b^1_i)_{x_i} + \sum (b^1_i)_w w_i \quad \text{if} \quad w_1 = c^1_1, \dots, w_n = c^1_n \tag{42}$$

equivalent to (32) is nontrivial. We conclude:

Theorem 3.1. *The exact inverse problem* (39) *admits a first–order solution f if and only if conditions* (40) *and* (41) *are satisfied. Then $f = f(\cdot\cdot, x_i, w, w_i, \cdot\cdot)$ is the unique solution of the initial problem*

$$\frac{\partial^2 f}{\partial w_i \partial w_{i'}} = F^{11}_{ii'}, f = b \text{ and } \frac{\partial f}{\partial w_i} = b^1_i \text{ if } w_1 = c^1_1, \dots, w_n = c^1_n \tag{43}$$

$(i, i' = 1, \dots, n)$. *Functions $c^1_1, \dots, c^1_n$ of variables $x_1, \dots, x_n, w$ are arbitrary and b^1_i, b may be arbitrary functions of the same variables satisfying* (42).

The result can be interpreted both in the local and in the global sense. A complementary result is as follows.

Theorem 3.2. *If an exact inverse problem with $m = 1$ and the second–order data*

$$e^1 = e^1[f] \qquad (e^1 = e^1(\cdot\cdot, x_i, w, w_i, w_{ii'}, \cdot\cdot)) \tag{44}$$

admits a solution, then the given function e^1 is in fact linear in the second–order variables and the problem admits even the first–order local solution.

Proof. It is of a mechanical nature: identity (7) immediately implies both the linearity and then the conditions (40) and (41). □

The subcase $m = 1$ is exceptional since the powerful contact geometry can be applied [2]. On this occasion, we recall the *absolute inverse problem*: to determine if a given system of differential equations is *equivalent* to the Euler–Lagrange system of a variational integral. In the favourable case $m = 1$, the equivalences are realized by merely a contact transformations, however, complete final results are not simple. For instance, the equation

$$F^1(\cdot\cdot, x_{i'}, w, w_{i'}, \cdot\cdot) - \sum w_{ii} = 0 \tag{45}$$

locally corresponds to a variational integral if and only if

$$F^1 = \frac{1}{2}\bar{b}_w(\cdot\cdot)\sum(w_i)^2 + \sum \bar{b}_{x_i}(\cdot\cdot)w_i + a(\cdot\cdot) \quad ((\cdot\cdot) = (x_1, \dots, x_n, w))$$

where $a, \bar{b}$ are arbitrary functions [2, Example 1 to Theorem 1.2].

On the other hand, the *exact* inverse problem is much easier. As for the equation (45), we obtain the conditions $(F^1)_{w_i} = 0$ by applying (41) and then the particular solution

$$f = \frac{1}{2}\sum(w_i)^2 + \int F^1 \mathrm{d}w \tag{46}$$

follows by choosing $b_i^1 = 0$ in (42) and (43). One can also easily obtain *all* local solutions. The *global solutions* are more involved. We recall the fibered space $\pi_0 : P = \mathbb{R}^{n+1} \to M = \mathbb{R}^n$ where $x_1, \dots, x_n, w$ are coordinates on P and $x_1, \dots, x_n$ are coordinates on M. Let F^1 be defined on a domain $\mathcal{B} \subset P$. The fibration π_0 can be applied on $\mathcal{B}$ and the following result is obvious.

Proposition 3.3. *If the fibration π_0 of $\mathcal{B}$ admits a global cross–section $\mathcal{C}$ with connected fibers $\mathcal{F}(C), C \in \mathcal{C}$, then every choice of the initial values on $\mathcal{C}$ for the integral $\int F^1\mathrm{d}w$ in formula* (46) *provides a global solution of the exact inverse problem for the equation* (45).

If there are not global sections $\mathcal{C}$ or the fibers $\mathcal{F}(C)$ are disconnected, the ramified (multiply–valued) solutions of the exact inverse problem may appear.

For the general exact inverse problem (39), still one global result is worth mentioning. Assume that only the functions $F_{ii'}^{jj'}$ satisfying (40) are given. Then the remaining function F^1 resolves the Pfaffian equation

$$d_1F^1 = -\sum\left((F_{ii'}^{11})_{x_i} + \sum(F_{ii'}^{11})_w w_i\right)\mathrm{d}w_i, \tag{47}$$

equivalent to (41). The local integrability condition $(d_1)^2F^1 = 0$ is satisfied and *global* solutions are affected by the cohomologies $H^1(\mathcal{F}(B))$.

4 The Subcase of One Independent Variable

We suppose $n = 1$. Let us abbreviate

$$x = x_1, w_2^j = w_{11}^j, w_3^j = w_{111}^j, \dots, f^j = f_1^j = f_{w_1^j}, f^{jj'} = f_{11}^{jj'} = f_{w_1^j w_1^{j'}}, \dots$$

but otherwise the notation is retained. Then

$$f = f(x, \cdot\cdot, w^j, w_1^j, \cdot\cdot), e[f] = \sum e^j[f]w_t^j, e^j[f] = E^j[f] - \sum E_{11}^{jj'}[f]w_2^{j'}$$

where

$$E_{11}^{jj'}[f] = f^{jj'},\ E^j[f] = f_{w^j} - (f^j)_x - \sum(f^j)_{w^{j'}}w_1^{j'}.$$

The exact inverse problem

$$F_{11}^{jj'} = E_{11}^{jj'}[f],\ F^j = E^j[f] \qquad (j, j' = 1, \dots, m) \tag{48}$$

again simplifies since $G_{11}^{jj'} = 0$. The *first requirement* (48) is solvable if and only if

$$(F_{11}^{jj'})_{w_1^{j''}} = (F_{11}^{jj''})_{w_1^{j'}} \qquad (j, j', j'' = 1, \dots, m) \tag{49}$$

where the symmetry $F_{11}^{jj'} = F_{11}^{j'j}$ is supposed. Passing to the *second requirement* (48), we recall the functions

$$F_1^{j'j} = (F^j)_{w_1^{j'}} + (F_{11}^{jj'})_x + \sum(F_{11}^{jj'})_{w^{j''}}w_1^{j''} \tag{50}$$

and identities (22) which read

$$F_1^{j'j} + F_1^{jj'} = 0, (F_1^{j'j})_{w_1^{j''}} = (F_{11}^{j'j''})_{w^j} - (F_{11}^{jj''})_{w^{j'}} \quad (j, j', j'' = 1, \ldots, m). \tag{51}$$

Conditions (51) can be alternatively regarded as a (compatible) system of differential equations for the functions $F_1^{j'j}$ and then the definition (50) may be interpreted as a (compatible) system for the functions F^j. Therefore, in a certain sense, the data F^j can be calculated from $F_{ii'}^{jj'}$ by two quadratures. We omit the proof here.

There are additional local solvability conditions

$$d_0 \sum F_1^{j'j} \mathrm{d}w^j \wedge \mathrm{d}w^{j'} = 0, \tag{52}$$

$$2d_0 \sum F^j \mathrm{d}w^j + \left(\frac{\partial}{\partial x} + \sum c_1^{j''} \frac{\partial}{\partial w_1^{j''}}\right) \sum F_1^{j'j} \mathrm{d}w^j \wedge \mathrm{d}w^{j'} = 0 \tag{53}$$

at the level set $w_1^1 = c_1^1, \ldots, w_1^m = c_1^m$ where $c_1^j = c_1^j(x)$ are arbitrary but fixed functions. We conclude:

Theorem 4.1. *The exact inverse problem* (48) *admits the first–order solution f if and only if the identities* (49)–(53) *are satisfied. Then $f = f(x, \cdot\cdot, w^j, w_1^j, \cdot\cdot)$ is the unique solution of the initial problem*

$$\frac{\partial^2 f}{\partial w_1^j \partial w_1^{j'}} = F_{ii'}^{jj'}; f = b \text{ and } \frac{\partial f}{\partial w_1^j} = b_1^j \text{ if } w_1^1 = c_1^1, \ldots, w_1^m = c_1^m \tag{54}$$

(j,j'=1,. . . ,m) where the initial values satisfy

$$d_0 \sum b_1^{j'} \mathrm{d}w^{j'} = \sum F_1^{j'j} \mathrm{d}w^j \wedge \mathrm{d}w^{j'}, \tag{55}$$

$$d_0 b = \sum \left(F^j + (\frac{\partial}{\partial x} + \sum c_1^{j'} \frac{\partial}{\partial w^{j'}}) b_1^j\right) \mathrm{d}w^j \tag{56}$$

at the level set $w_1^1 = c_1^1, \ldots, w_1^m = c_1^m$.

The result can be again interpreted both in the local and in the global sense. The obstacles for the existence of the global and single–valued solution lie in the homology $H^1(\mathcal{F}(B))$ for the system (54) and in the cohomologies $H^2(\mathcal{F}(C))$ and $H^1(\mathcal{F}(C))$ for the conditions (55). Since functions w_i^j $(i = 1, \ldots, n; j = 1, \ldots, m)$ serve for coordinates on the fibers $\mathcal{F}(B)$ and quite other fuctions w^j $(j = 1, \ldots, m)$ provide coordinates on the fibers $\mathcal{F}(C)$, such nontrivial homologies with ramified solutions can be easily ilustrated by examples. For this aim, let us mention the inverse problem for the system of equations

$$F^j(x, \cdot\cdot, w^{j'}, w_1^{j'}, \cdot\cdot) - w_2^j = 0 \qquad (j = 1, \ldots, m) \tag{57}$$

which is a certain counterpart to the previous equation (45). Then the second identity (51) together with the definition equation (50) imply that

$$(F_1^{jj'})_{w^{j''}} = 0,\ F_1^{jj'} = (F^j)_{w_1^{j'}} \quad \text{hence} \quad F^j = \sum F_1^{jj'} w_1^{j'} + G^j.$$

The functions

$$F_1^{jj'} = F_1^{jj'}(x, \cdot\cdot, w^{j''}, \cdot\cdot),\ G^j = G^j(x, \cdot\cdot, w^{j'}, \cdot\cdot) \tag{58}$$

are subject only to the skew–symmetry $F_1^{jj'} = -F_1^{j'j}$ and local solvability conditions (52) and (53). Since they are otherwise quite arbitrary, it follows that the nontrivial cohomologies $H^2(\mathcal{F}(C))$ and $H^1(\mathcal{F}(C))$ related to the *global equations* (55) and (56) can be realized by an appropriate choice of functions (58). One can even suppose $F_1^{jj'} = b_1^j = 0$ identically hence $F^j = G^j$ for the equation (56), hence for the example with the nonvanishing cohomology $H^1(\mathcal{F}(C))$.

It moreover follows that the homological obstruction for the existence of global "economical" solution is more restrictive than the Takens' obstruction [9] for the solutions of arbitrary order.

The complementary result is strong.

Theorem 4.2. *If an exact inverse problem with the second–order data*

$$e^j = e^j[f] \qquad (j = 1, \ldots, m;\ e^j = e^j(x, \cdot\cdot, w^{j'}, w_1^{j'}, w_2^{j'}, \cdot\cdot))$$

admits a solution $f = f(\cdot\cdot, x_i, w_r^j, \cdot\cdot)$ *of any order, then the given functions* e^j *are in fact linear in the second–order variables and the exact inverse problem admits even the first–order local solution.*

Proof. Identity (7) in the extended jet space reads

$$\sum (e^{j'})_{w^j} w_t^{j'} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum (e^{j'})_{w_1^j} w_t^{j'}\right) - \frac{\mathrm{d}}{\mathrm{d}t} e^j + \frac{d^2}{\mathrm{d}x^2}\left(\sum (e^{j'})_{w_2^j} w_t^{j'}\right) = 0.$$

The linearity in w_2^j immediately follows, however, our solvability conditions cannot be easily derived from the Helmholz conditions by merely a formal calculus. The complete proof rests on the *reduction principle* [12, Theorem 4.5.5]: *If* $n = 1$ *then every Euler–Lagrange system of even order* $2K$ *locally corresponds to an appropriate Lagrange function of order* K. In our case $K = 1$. See the Appendix below for a new elementary proof. □

Owing to the reduction principle, the subcase $n = 1$ is exceptional, too. Even the solution of the *general inverse problem* becomes easier since only the algebraical adaptations of the systems (the multipliers) are sufficient *in the nondegenerate case*. The famed article [4] may be instructive in this respect: the main effort is focused on the toilsome *compatibility investigations*, not on the marginal geometrical aspects.

The local exact inverse problem with $n = 1$ can be easily included in the common jet theory. Alas, the claims in the reduction method [17] should be taken with a certain caution: for instance, the forceful decomposition $L = -T + V$ [17, Theorem 3.11] of a Lagrange function L into the "kinetic energy" and the "potential function" in fact depends on the method of the calculation and does not make any proper geometrical sense.

5 The Subcase of Two Dependent Variables

We suppose $m = 2$ with the range of indices $i, i', i'' = 1, \ldots, n$ ($n \geq 2$) and $j, j', j'' = 1, 2$. The original notation is preserved. The given functions

$$F_{ii'}^{jj'} = F_{i'i}^{jj'} = F_{i'i}^{j'j} = F_{ii'}^{j'j},\ F^j \tag{59}$$

are of the symmetrical nature while the auxiliary functions

$$G_{ii'}^{jj} = G_{ii}^{jj'} = 0,\ G_{ii'}^{12} = -G_{i'i}^{12} = G_{i'i}^{21} = -G_{ii'}^{21} \tag{60}$$

are of the skew–symmetric kind and need not identically vanish. This fact makes the exact inverse problem nontrivial.

Let us recall the main achievements of Section 2.

The *first requirement* (16) was clarified by equations (19) which read

$$\left(F_{ii'}^{jj}\right)_{w_{i''}^j} = \left(F_{ii''}^{jj}\right)_{w_{i'}^j},\ \left(F_{ii}^{12}\right)_{w_{i'}^1} = \left(F_{ii'}^{11}\right)_{w_i^2},\ \left(F_{ii}^{21}\right)_{w_{i'}^2} = \left(F_{ii'}^{22}\right)_{w_i^1}, \tag{61}$$

$$\left(F_{ii'}^{12} + G_{ii'}^{12}\right)_{w_{i''}^1} = \left(F_{ii''}^{11}\right)_{w_{i'}^2},\ \left(F_{ii'}^{21} + G_{ii'}^{21}\right)_{w_{i''}^2} = \left(F_{ii''}^{22}\right)_{w_{i'}^1} \quad (i \neq i') \tag{62}$$

in our case $m = 2$. Identities (61) concern only the given functions (59) while identities (62) may be regarded as differential equations for the functions (60).

The *second requirement* (16) was represented by identities (22) completed with the solvability conditions (31) and (34). The first identity (22) reads

$$\left(F^j\right)_{w_{i'}^{j'}} + \left(F^{j'}\right)_{w_{i'}^j} + 2\sum \left(F_{ii'}^{jj'}\right)_{x_i} + 2\sum \left(F_{ii'}^{jj'}\right)_{w^{j''}} w_i^{j''} = 0, \tag{63}$$

by using definition (21) and the skew–symmetry (60). It concerns only the functions (59). The second identity (22) is trivial if $j = j'$. Assuming $j \neq j'$, we obtain only the equations

$$\begin{aligned} \left(F^{12}_{i'}\right)_{w^1_i} &= \left(F^{11}_{i'i}\right)_{w^2} - \left(F^{21}_{i'i} + G^{21}_{i'i}\right)_{w^1} , \\ \left(F^{12}_{i'}\right)_{w^2_i} &= \left(F^{12}_{i'i} + G^{12}_{i'i}\right)_{w^2} - \left(F^{22}_{i'i}\right)_{w^1} \end{aligned} \tag{64}$$

which simplify if $i = i'$. Condition (31) is trivial and condition (34) reads

$$\left(F^1\right)_{w^2} - \left(F^2\right)_{w^1} + \sum \left(\frac{\partial}{\partial x_i} + \sum c^j_i \frac{\partial}{\partial w^j}\right) F^{12}_i = 0 \tag{65}$$

if $w^1_1 = c^1_1, \ldots, w^2_n = c^2_n$. Functions F^{12}_i appearing here are defined in (21).

In more detail, on this occasion, the lower indices i, i', i'' are completed with additional $k, k' = 1, \ldots, n$ for aesthetic reasons.

Lemma 5.1. *The skew–symmetry* (60) *is ensured if and only if identities*

$$\left(F^{11}_{ik}\right)_{w^2_{i'}} + \left(F^{11}_{i'k}\right)_{w^2_i} = 2\left(F^{12}_{ii'}\right)_{w^1_k} , \ \left(F^{22}_{ik}\right)_{w^1_{i'}} + \left(F^{22}_{i'k}\right)_{w^1_i} = 2\left(F^{12}_{ii'}\right)_{w^2_k} \tag{66}$$

are satisfied.

Proof. We recall that (62) is regarded as the system of differential equations

$$\frac{\partial G^{12}_{ii'}}{\partial w^1_k} = \left(F^{11}_{ik}\right)_{w^2_{i'}} - \left(F^{12}_{ii'}\right)_{w^1_k} , \ \frac{\partial G^{21}_{ii'}}{\partial w^2_k} = \left(F^{22}_{ik}\right)_{w^1_{i'}} - \left(F^{12}_{ii'}\right)_{w^2_k} . \tag{67}$$

The skew–symmetry (60) holds true if and only if the system

$$\frac{\partial G^{12}_{ii'}}{\partial w^1_k} = -\left(F^{11}_{i'k}\right)_{w^2_i} + \left(F^{12}_{i'i}\right)_{w^1_k} , \ \frac{\partial G^{21}_{ii'}}{\partial w^2_k} = -\left(F^{22}_{i'k}\right)_{w^1_i} - \left(F^{21}_{i'i}\right)_{w^2_k}$$

appearing by the exchange $i \leftrightarrow i'$ is identical with (67). This observation immediately implies the assertion of Lemma 5.1. □

Lemma 5.2. *Assuming the skew–symmetry, system* (67) *is compatible if and only if all identities*

$$\begin{gathered} \left(F^{11}_{ik}\right)_{w^2_{i'}w^2_{k'}} + \left(F^{22}_{ik'}\right)_{w^1_{i'}w^1_k} = 2\left(F^{12}_{ii'}\right)_{w^2_{k'}w^1_k} , \\ \left(F^{11}_{ik}\right)_{w^1_{i'}w^2_{k'}} = \left(F^{11}_{ii'}\right)_{w^2_{k'}w^1_k} , \ \left(F^{22}_{ik}\right)_{w^2_{i'}w^1_{k'}} = \left(F^{22}_{ii'}\right)_{w^1_{k'}w^2_k} \end{gathered} \tag{68}$$

are satisfied.

Proof. Routine application of the rule

$$\left(G^{12}_{ii'}\right)_{w^1_k w^2_{k'}} = \left(G^{12}_{ii'}\right)_{w^2_{k'}w^1_k} , \ \left(G^{12}_{ii'}\right)_{w^j_k w^j_{k'}} = \left(G^{12}_{ii'}\right)_{w^j_{k'}w^j_k}$$

to the equation (67) is combined with the skew–symmetry. For instance, we have

$$(G^{12}_{ii'})_{w^1_k w^1_{k'}} = \Big((F^{11}_{ik})_{w^2_{i'}} - (F^{12}_{ii'})_{w^1_k}\Big)_{w^1_{k'}} , (G^{12}_{ii'})_{w^1_{k'}w^1_k} = \Big((F^{11}_{ik'})_{w^2_{i'}} - (F^{12}_{ii'})_{w^1_{k'}}\Big)_{w^1_k}$$

and therefore

$$\left(F^{11}_{ik}\right)_{w^2_{i'}w^1_{k'}} = \left(F^{11}_{ik'}\right)_{w^2_{i'}w^1_k}$$

which provides the middle equation (68). □

We conclude that the system (67) admits a certain general skew–symmetrical solution

$$G^{12}_{ii'} = \bar{G}^{12}_{ii'} + C_{ii'} \quad (C_{ii'} = C_{ii'}(x_1, \ldots, x_n, w^1, w^2), C_{ii'} = -C_{i'i}) \tag{69}$$

where $\bar{G}^{12}_{ii'}$ is the particular solution of the same system (67) such that

$$\bar{G}^{12}_{ii'} = 0 \quad \text{if} \quad w^1_1 = c^1_1, \ldots, w^2_n = c^2_n \quad (c^j_i = c^j_i(x_1, \ldots, x_n, w^1, w^2)). \tag{70}$$

Functions $C_{ii'}$ and c^j_i are arbitrary.

Lemma 5.3. *Equations* (64) *can be expressed only in terms of functions* (59).

Proof. Recalling the particular case

$$F^{12}_{i'} = (F^2)_{w^1_{i'}} + \sum (F^{21}_{ki'} + G^{21}_{ki'})_{x_k} + \sum (F^{21}_{ki'} + G^{21}_{ki'})_{w^{j''}} w^{j''}_k$$

of definition (21), let us deal with the first equation (64). We are interested only in the occurences of functions (60). So we have the equation

$$\cdots + \sum (G^{21}_{ki'})_{x_k w^1_i} + \sum ((G^{21}_{ki'})_{w^{j''}} w^{j''}_k)_{w^1_i} = \cdots - (G^{21}_{i'i})_{w^1} .$$

However

$$(G^{21}_{ki'})_{x_k w^1_i} = (G^{21}_{ki'})_{w^1_i x_k} = \left(-(F^{11}_{ki})_{w^2_{i'}} + (F^{12}_{ii'})_{w^1_k}\right)_{x_k},$$

$$((G^{21}_{ki'})_{w^{j''}} w^{j''}_k)_{w^1_i} = \left(-(F^{11}_{ki})_{w^2_{i'}} + (F^{12}_{ii'})_{w^1_k}\right)_{w^{j''}} w^{j''}_k + (G^{21}_{ii'})_{w^1}$$

with the use of the first equation (67). We are done since $G^{21}_{ii'} = -G^{21}_{i'i}$. The second equation (64) is analogous. □

Let us finish with the remaining condition (65). If functions

$$F^{12}_i = \left(F^2\right)_{w^1_i} + \sum \left(F^{21}_{ki} + G^{21}_{ki}\right)_{x_k} + \sum \left(F^{21}_{ki} + G^{21}_{ki}\right)_{w^{j''}} w^{j''}_k$$

with (69) employed are inserted into (65), we obtain

$$(F^1)_{w^2} - (F^2)_{w^1} =$$

$$= \sum (\frac{\partial}{\partial x_i} + \sum c^j_i \frac{\partial}{\partial w^j}) \left((F^2)_{w^1_i} + \sum (F^{21}_{ki} + C_{ki})_{x_k} + \sum (F^{21}_{ki} + C_{ki})_{w^{j''}} c^{j''}_k\right) =$$

$$= \sum (\frac{\partial}{\partial x_i} + \sum c^j_i \frac{\partial}{\partial w^j}) \left((F^2)_{w^1_i} + \sum (F^{21}_{ki})_{x_k} + \sum (F^{21}_{ki})_{w^{j''}} c^{j''}_k\right)$$

owing to the skew–symmetry $C_{ki} = -C_{ik}$. It follows that

$$2\left(\left(F^1\right)_{w^2} - \left(F^2\right)_{w^1}\right) = \sum (\frac{\partial}{\partial x_i} + \sum c^j_i \frac{\partial}{\partial w^j}) \left((F^2)_{w^1_i} - (F^1)_{w^2_i}\right) \tag{71}$$

if moreover identity (63) is applied. Condition (71) is considered only at the level set $w^j_i = c^j_i = c^j_i(x_1, \ldots, x_n)$, see (34).

We can eventually summarize as follows.

Theorem 5.1 (solvability)**.** *Identities* (61), (63), (64) *with Lemma* 5.3 *applied, together with* (66), (68) *and* (71) *provide necessary and sufficient solvability conditions for the given data* (59) *of the exact inverse problem* (16) *with* $m = 2$.

Theorem 5.2 (auxiliary functions)**.** *Functions* (60) *are skew–symmetrical solutions of the system* (67). *They are represented by formulae* (69) *and* (70).

Theorem 5.3 (the solution)**.** *The solution* f *of the exact inverse problem* (16) *is given by formula* (29) *where* $\bar{f}$ *is a particular solution of system* (27) *satisfying* (28) *and the coefficients* b^j_i, b *are given by equations* (30) *and* (32) *for the particular case* $m = 2$.

6 The Subcase of Two Independent Variables

We suppose $n = 2$ and $m \geq 2$. The given functions

$$F_{ii'}^{jj'} = F_{i'i}^{jj'} = F_{i'i}^{j'j} = F_{ii'}^{j'j}, F^i \quad (i, i' = 1, 2;\ j, j' = 1, \ldots, m) \tag{72}$$

of the symmetrical nature and the skew–symmetrical auxiliary functions

$$G_{ii'}^{jj} = G_{ii}^{jj'} = 0,\ G_{12}^{jj'} = -G_{21}^{jj'} = G_{21}^{j'j} = -G_{12}^{j'j} \quad (i, i' = 1, 2) \tag{73}$$

formally differ from (59) and (72) by merely an exchange of the role of the upper and the lower indices. We therefore simulate the previous subcase $m = 2$.

The *first requirement* (16) reads

$$(F_{ii}^{jj'})_{w_i^{j''}} = (F_{ii}^{jj''})_{w_i^{j'}},\ (F_{12}^{jj})_{w_1^{j'}} = (F_{11}^{jj'})_{w_2^j},\ (F_{21}^{jj})_{w_2^{j'}} = (F_{22}^{jj'})_{w_1^j}, \tag{74}$$

$$(F_{12}^{jj'} + G_{12}^{jj'})_{w_1^{j''}} = (F_{11}^{jj''})_{w_2^{j'}},\ (F_{21}^{jj'} + G_{21}^{jj'})_{w_2^{j''}} = (F_{22}^{jj''})_{w_1^{j'}} \tag{75}$$

and the first identity (22) of the *second requirement* (16) reads

$$(F^j)_{w_{i'}^{j'}} + (F^{j'})_{w_{i'}^j} + 2\sum(F_{ii'}^{jj'})_{x_i} + 2\sum(F_{ii'}^{jj'})_{w^{j''}} w_i^{j''} = 0 \tag{76}$$

which is formally the same as (63). However the second identity (22) reads

$$(F_{i'}^{j'j})_{w_i^k} = (F_{i'i}^{j'k} + G_{i'i}^{j'k})_{w^j} - (F_{i'i}^{jk} + G_{i'i}^{jk})_{w^j} \tag{77}$$

and slightly differs from (64), moreover the condition (31) is nontrivial if $m \geq 3$ and there are several equations

$$(F^j)_{w^{j'}} - (F^{j'})_{w^j} + \sum(\frac{\partial}{\partial x_i} + \sum c_i^{j''} \frac{\partial}{\partial w^{j''}})F_i^{jj'} = 0 \quad (j, j' = 1, \ldots, m) \tag{78}$$

instead of the single condition (65).

Lemma 6.1. *The skew–symmetry is ensured if and only if*

$$(F_{11}^{jk})_{w_2^{j'}} + (F_{11}^{j'k})_{w_2^j} = 2(F_{12}^{jj'})_{w_1^k}, (F_{22}^{jk})_{w_1^{j'}} + (F_{22}^{j'k})_{w_1^j} = (F_{12}^{jj'})_{w_2^k}. \tag{79}$$

Proof. The same method as above should be applied to the system

$$\frac{\partial G_{12}^{jj'}}{\partial w_1^{j''}} = (F_{11}^{jj''})_{w_2^{j'}} - (F_{12}^{jj'})_{w_1^{j''}},\ \frac{\partial G_{21}^{jj'}}{\partial w_2^{j''}} = (F_{22}^{jj''})_{w_1^{j'}} - (F_{21}^{jj'})_{w_2^{j''}}. \tag{80}$$

□

Lemma 6.2. *Assuming the skew–symmetry, identities*

$$\begin{gathered}(F_{11}^{jk})_{w_2^{j'} w_2^{k'}} + (F_{22}^{jk'})_{w_1^{j'} w_1^k} = 2(F_{12}^{jj'})_{w_2^{k'} w_1^k},\\ (F_{11}^{jk})_{w_1^{j'} w_2^{k'}} = (F_{11}^{jj'})_{w_2^{k'} w_1^k},\ (F_{22}^{jk})_{w_2^{j'} w_1^{k'}} = (F_{22}^{jj'})_{w_1^{k'} w_2^k}\end{gathered} \tag{81}$$

ensure the compatibility of equations (80).

Lemma 6.3. *Equations* (77) *can be expressed in terms of functions* (72).

The proofs may be omitted.

Altogether we conclude that the system (80) admits a certain general skew–symmetrical solution

$$G_{12}^{jj'} = \bar{G}_{12}^{jj'} + C^{jj'} \quad (C^{jj'} = C^{jj'}(x_1, x_2, w^1, \ldots, w^m), C^{jj'} = -C^{j'j}) \tag{82}$$

where $\bar{G}_{12}^{jj'}$ is the particular solution of (80) such that

$$\bar{G}_{12}^{jj'} = 0 \quad \text{if} \quad w_1^1 = c_1^1, \ldots, w_2^m = c_2^m \tag{83}$$

and $c_i^j = c_i^j(x_1, x_2, w^1, \ldots, w^m), C^{jj'}$ are arbitrary functions.

Let us continue with the remaining conditions (31) and (78). They are more involved. The condition (78) is easier. If functions $F_{i'}^{j'j}$ $(i' = 1, 2)$ and (82) are inverted into (78), we obtain the conditions

$$2\left((F^j)_{w^{j'}} - (F^{j'})_{w^j}\right) = \sum(\frac{\partial}{\partial x_i} + \sum c_i^{j''}\frac{\partial}{\partial w^{j''}})\left((F^{j'})_{w_i^j} - (F^j)_{w_i^{j'}}\right) \tag{84}$$

quite analogously as above.

We consider the remaining condition (31). It is equivalent to the "cyclic" system of equations

$$(F_i^{jj'})_{w^{j''}} + (F_i^{j'j''})_{w^j} + (F_i^{j''j})_{w^{j'}} = 0 \quad (i, i' = 1, 2;\ j, j', j'' = 1, \ldots, m). \tag{85}$$

Inserting (21) with the use of (82) and (83), one can obtain equivalent system

$$(\frac{\partial}{\partial x_i} + \sum c_i^k \frac{\partial}{\partial w^k})\mathcal{G}^{jj'j''} = \mathcal{F}^{jj'j''} \quad (i = 1, 2;\ j, j', j'' = 1, \ldots, m) \tag{86}$$

with the "cyclic" sum

$$\mathcal{G}^{jj'j''} = (C^{jj'})_{w^{j''}} + (C^{j'j''})_{w^j} + (C^{jj''})_{w^{j'}} \quad (j, j', j'' = 1, \ldots, m) \tag{87}$$

where the right–hand side $\mathcal{F}^{jj'j''}$ is expressed only in terms of the given functions (72) and need not be explicitly stated here. We may assume $j \neq j' \neq j''$ and even $j < j' < j''$ without any loss of generality. For instance if $m = 3$, we have only two equations (86) with three unknown functions C^{12}, C^{13} and C^{23}. The number of equations (86) exceeds the number of unknown functions $C^{jj'}$ if $m > 3$.

Let us attempt the summary.

Theorem 6.1 (solvability)**.** *Identities* (74), (76), (77) *with Lemma 6.3 applied, together with* (79), (81), (84) *and conditions following from* (86) *provide necessary and sufficient solvability conditions of the exact inverse problem* (16) *with* $n = 2$.

Theorem 6.2 (auxiliary functions)**.** *Functions* (73) *are skew-symmetrical solutions of the system* (80). *They are represented by formulae* (82) *and* (83) *where the "integration constants* $C^{jj'}$*" are subjected to conditions* (86).

Theorem 6.3 (the solution)**.** *Literally the same as Theorem 5.3.*

We believe that both subcases $m = 2$ and $n = 2$ provide much better insight into the proper nature both of the local and of the global exact inverse problem than the common Helmholz theory, however, the alternative coordinate–free geometrical approach would be highly desirable, too.

7 The Decomposed Subcase

The formal novelty of our method lies in the auxiliary functions $G_{ii'}^{jj'}$ which ensure the compatibility (19) of differential equations (37) for the unknown function f. In general, the existence of functions $G_{ii'}^{jj'}$ is a delicate problem, however, there are some exceptions. Let us suppose

$$\begin{array}{ll} F_{ii'}^{jj'} = F_{ii'}^{jj'}(\cdot\cdot, x_{i''}, w^{j''}, \cdot\cdot) & \text{if} \quad j \neq j' \quad \text{and} \quad i \neq i', \\ F_{ii'}^{jj} = F_{ii'}^{jj}(\cdot\cdot, x_{i''}, w^{j''}, w_{i''}^j, \cdot\cdot) & \text{if} \quad i \neq i', \\ F_{ii}^{jj'} = F_{ii}^{jj'}(\cdot\cdot, x_{i''}, w^{j''}, w_i^{j''}, \cdot\cdot) & \text{if} \quad j \neq j', \\ F_{ii}^{jj} = F_{ii}^{jj}(\cdot\cdot, x_{i''}, w^{j''}, w_i^{j'}, w_{i'}^j, \cdot\cdot) & \end{array} \tag{88}$$

where $i, i', i'' = 1, \ldots, n$ and $j, j', j'' = 1, \ldots, m$. Then the *first requirement* (19) separately concerns either only the given functions or the auxiliary functions. In more detail, we have two autonomous systems of conditions, namely

$$\begin{gathered}(F^{jj}_{ii'})_{w^j_{i''}} = (F^{jj}_{ii''})_{w^j_{i'}},\ (F^{jj'}_{ii})_{w^{j''}_i} = (F^{jj''}_{ii})_{w^{j'}_i} \\ (F^{jj'}_{ii'})_{w^{j''}_{i''}} = 0 \text{ if } i \neq i' \text{ and } j \neq j'\end{gathered} \tag{89}$$

for the given functions $F^{jj'}_{ii'}$ and

$$(G^{jj'}_{ii'})_{w^{j''}_{i''}} = (G^{jj''}_{ii''})_{w^{j'}_{i'}}, \tag{90}$$

for the unknown functions $G^{jj'}_{ii'}$. Complete solution of the first requirement (19) becomes possible. First of all, regardless of functions $G^{jj'}_{ii'}$ and assuming only the identities (89), the system

$$\frac{\partial^2 F}{\partial w^j_i \partial w^{j'}_{i'}} = F^{jj'}_{ii'} \quad (i, i' = 1, \ldots, n;\ j, j' = 1, \ldots, m) \tag{91}$$

is compatible. Second, one can observe that the condition (90) implies

$$(G^{jj'}_{ii'})_{w^{j_1}_{i_1} \cdots w^{j_k}_{i_k}} = (G^{j_1 j_2}_{i_1 i_2})_{w^j_i w^{j'}_{i'} w^{j_3}_{i_3} \cdots w^{j_k}_{i_k}} \tag{92}$$

and it follows that functions $G^{jj'}_{ii'}$ are polynomials in variables $w^1_1, \ldots, w^m_n$ of the order $\min(m, n)$ at most. (Hint: if $j_1 = j_2$ or $i_1 = i_2$ then the right–hand side, hence the left–hand side of (92), vanish.) If $g = g(\cdot\cdot, x_i, w^j, w^j_i, \cdot\cdot)$ is *any* function satisfying

$$\frac{\partial^2 g}{\partial w^j_i \partial w^{j'}_{i'}} = G^{jj'}_{ii'} \quad (i, i' = 1, \ldots, n;\ j, j' = 1, \ldots, m) \tag{93}$$

then g is such a polynomial too (by applying the same argument) and therefore

$$g = \sum g^{j_1 \cdots j_k}_{i_1 \cdots i_k}(\cdot\cdot, x_i, w^j, \cdot\cdot) \det \begin{pmatrix} w^{j_1}_{i_1} & \cdots & w^{j_1}_{i_k} \\ \cdots & & \cdots \\ w^{j_k}_{i_1} & \cdots & w^{j_k}_{i_k} \end{pmatrix}, \tag{94}$$

where

$$i_1 < \cdots < i_k;\ j_1 < \cdots < j_k;\ 0 \leq k \leq \min(m, n).$$

Conversely any function (94) satisfies the system (93) with appropriate $G^{jj'}_{ii'}$ (which need not be explicitly stated here). Altogether we conclude that the *general solution* of the system

$$\frac{\partial^2 f}{\partial w^j_i \partial w^{j'}_{i'}} = F^{jj'}_{ii'} + G^{jj'}_{ii'} \quad (i, i' = 1, \ldots, n;\ j, j' = 1, \ldots, m) \tag{95}$$

is $f = F + g$, the sum of a particular solution F of (91) and the general solution g of (93). The first requirement is completely resolved.

With this result, the second requirement (16) can be rather easily investigated. Indeed, if f denotes the sought solution of the exact inverse problem (16) then

$$\begin{gathered}E^{jj'}_{ii'}[f - F] = E^{jj'}_{ii'}[f] - E^{jj'}_{ii'}[F] = 0, \\ E^j[f - F] = E^j[f] - E^j[F] = F^j - E^j[F].\end{gathered}$$

It follows that the above function $g = f - F$ is a solution of the first–order exact inverse problem

$$0 = E^{jj'}_{ii'}[g],\ (G^j =) F^j - E^j[F] = E^j[g] = e^j[g] \tag{96}$$

and such a solution g is available by inserting the formula (94) into the second requirement $G = e^j[g]$ of the reduced inverse problem (96).

Theorem 7.1. *Assuming* (88), *solution of the exact inverse problem can be decomposed as* $f = F + g$. *Any particular solution of the system* (91) *can be taken for the function* F *and function* g *given by* (94) *resolves the exact inverse problem* (96).

Both the exceptional subcases $m = 1$ and $n = 1$ and also the first–order exact variational problems where $F^{jj'}_{ii'} = 0$ identically extensively treated in [16] are involved in Theorem 7.1.

Appendix

We temporarily suppose $n = 1$ with the same alternative notation of the jet coordinates

$$x = x_1, w^j_r = w^j_{1\cdots 1} \qquad (r \text{ terms } 1\cdots 1;\ j = 1,\dots,m)$$

as in Section 4 above. Recalling the Lagrange functions and the Euler–Lagrange expressions

$$f = f(x,\cdot\cdot,w^j_r,\cdot\cdot), \quad e^j[f] = \sum(-1)^r \frac{d^r}{\mathrm{d}x^r}\frac{\partial f}{\partial w^j_r} \quad (j = 1,\dots,m),$$

we can state short proof of the following result.

Proposition A1 (local). *Let the Euler–Lagrange expressions* $e^j[f]$ $(j = 1,\dots,m)$ *be of the order* S *(at most). If* $S = 2K$ *is even then* $e^j[f] = e^j[g]$ *for an appropriate Lagrange function* g *of the order* K. *If* $S = 2K + 1$ *is odd then*

$$e^j[f] = e^j[g_0 + \sum g_k w^k_{K+1}]$$

where $g_0,\dots,g_m$ *are appropriate functions of the order* K.

The original proof [12, p. 56–68] rests on the use of the Poincaré–Cartan forms which obscures the elementary nature of the result. The "naive" approach is much shorter. Indeed, let f be the Lagrange function just of the order K. Then the higher–order terms of the Euler–Lagrange expressions $e^j[f]$ are

$$\begin{array}{lll} (-1)^K \sum f_{w^j_K w^{j'}_K} w^{j'}_{2K} & (f = f(x,\dots,w^m_K)), & (A) \\ (-1)^K \sum (f^j_{w^{j'}_{K-1}} - f^{j'}_{w^j_{K-1}}) w^{j'}_{2K-1} & (f = \cdot\cdot + \sum f^j(x,\dots,w^m_{K-1}) w^j_K) & (B) \end{array}$$

according to whether f is nonlinear or linear in the top–order variables. We have the order equalities

$$\begin{array}{ll} \max \text{ order } e^j[f] = 2\,\text{order}\, f & \text{in } (A) \\ \max \text{ order } e^j[f] = 2\,\text{order}\, f - 1 & \text{in } (B) \end{array}$$

except for the case when

$$\frac{\partial f^j}{\partial w^{j'}_{K-1}} = \frac{\partial f^{j'}}{\partial w^j_{K-1}} \quad (j, j' = 1,\dots,m), \quad f^j = \frac{\partial F}{\partial w^j_{K-1}}$$

for appropriate function $F = F(x,\dots,w^m_{K-1})$. However then f in (B) can be replaced with the Lagrange function

$$\bar{f} = f - \frac{\mathrm{d}}{\mathrm{d}x}F \quad \left(e^j[f] = e^j[\bar{f}];\ j = 1,\dots,m\right)$$

of the lower order. We conclude that there does exist the Lagrange function exactly satisfying both the above order equalities. This is just the Proposition.

Assuming $n > 1$ from now on, then analogous order reduction is possible but the investigations cause serious difficulties. To our best knowledge, we can mention only the following result.

Proposition A2 (local). *Let* $e^j[f]$ $(j = 1,\dots,m)$ *be the second–order Euler–Lagrange expressions linear in the second–order variables. Then* $e^j[f] = e^j[g]$ *for an appropriate first–order Lagrange function* g.

This is a generalization of easy Theorem 3.2 if $m > 1$. The linearity is *postulated* here. The Proposition A2 ensures that the Helmholz condition (7) together with the second–order linearity is sufficient for the *local* solvability of the inverse problem (16). It should be however noted that the original proof [6, p. 796–804] consists of lengthy technical corrections of the Tonti solution. However, assuming certain formula

$$e^j[F] = F^j - \sum F^{jj'}_{ii'} w^{j'}_{ii'}$$

with *given* functions $F^j, F^{jj'}_{ii'}$ but unknown f, we have $e^j[f] = e^j[\tilde{f}]$ where the Tonti integral (11) is of the second–order and (clearly) linear in the second–order variables:

$$\tilde{f} = \cdots + \sum b^j_{ii'} w^j_{ii'} \qquad (b^j_{ii'} = b^j_{i'i}).$$

Our aim is to determine a correction

$$c = \sum \frac{\mathrm{d}}{\mathrm{d}x_i} c_i = \cdots + \sum \left((c_i)_{w^j_{i'}} + (c_{i'})_{w^j_i} \right) w^j_{ii'}$$

in order to obtain the first–order function $g = \tilde{f} - c$. We obtain the system

$$(c_i)_{w^j_{i'}} + (c_{i'})_{w^j_i} = b^j_{ii'} \qquad (i, i' = 1, \ldots, n; j = 1, \ldots, m)$$

for the unknown first–order functions c_i. In fact we have m separate systems, each for every fixed $j = 1, \ldots, m$. Roughly saying, we may suppose $m = 1$, however, then Theorem 3.2 can be applied: the correction exists and we are done.

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