Report on the Absolute Differential Equations I

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Abstract The article provides a survey of the absolute theory of general systems of (partial) differential equations. The equations are relieved of all additional structures and subject to quite arbitrary change of the variables. An abstract mathematical theory in the Bourbaki sense with its own concepts and technical tools follows. In particular the external, internal, generalized and higher–order symmetries and infinitesimal symmetries together with the E. Cartan's prolongations, various characteristics, the involutivity and the controllability structures are clarified in genuinely coordinate–free terms without any use of the common jet mechanisms.

Keywords: Higher–order transformations; symmetry; diffiety; involutivity; controllability; characteristics.

1 Preface

We motivate the central concepts and informally describe the main task in order to give some impressions on our subject, however, this is not a logical prerequisite for the text to follow.

1.1 The Higher–order Transformations (the Morphisms)

Let us recall the *jet coordinates*

$$x_i, w_I^j \quad (j = 1, \ldots, m; I = i_1 \cdots i_r; i, i_1, \ldots, i_r = 1, \ldots, n; r = 0, 1, \ldots) \tag{1}$$

where x_i are *independent variables*, w^j (empty I) *dependent variables* and w_I^j (nonempty I) correspond to the *derivatives*

$$w_I^j = \frac{\partial w^j}{\partial x_I} = \frac{\partial^r w^j}{\partial x_{i_1} \cdots \partial x_{i_r}} \quad (I = i_1 \cdots i_r).$$

Let us moreover introduce the equations

$$\bar{x}_i = X_i(\cdot\cdot, x_{i'}, w_{I'}^{j'}, \cdot\cdot), \bar{w}^j = W^j(\cdot\cdot, x_{i'}, w_{I'}^{j'}, \cdot\cdot) \quad (i = 1, \ldots, n; j = 1, \ldots, m) \tag{2}$$

where X_i and W^j are given functions of a finite number of variables (1). They are interpreted as the transformation formulae: the functions

$$w^j = w^j(x_1, \ldots, x_n); \ j = 1, \ldots, m;$$

are transformed into certain functions

$$\bar{w}^j = \bar{w}^j(\bar{x}_1, \ldots, \bar{x}_n); \ j = 1, \ldots, m;$$

and this is made as follows. Denoting

$$\mathcal{X}_i = X_i(\cdot\cdot, x_{i'}, \frac{\partial w^{j'}}{\partial x_{I'}}(x_1, \ldots, x_n), \cdot\cdot) = \mathcal{X}_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n),$$

we suppose that

$$\det\left(\frac{\partial \mathcal{X}_{i'}}{\partial x_i}\right) = \det(D_i\mathcal{X}_{i'}) \neq 0 \qquad (D_i = \frac{\partial}{\partial x_i} + \sum w_{Ii}^j \frac{\partial}{\partial w_I^j}).$$

Then the implicit system $\bar{x}_i = \mathcal{X}_i(x_1, \ldots, x_n); i = 1, \ldots, n$; admits a solution $x_i = \bar{\mathcal{X}}_i(\bar{x}_1, \ldots, \bar{x}_n)$; $i = 1, \ldots, n$; and provides the desired result

$$\bar{w}^j = W^j(\cdot\cdot, \bar{\mathcal{X}}_i, \frac{\partial w^{j'}}{\partial x_{I'}}(\bar{\mathcal{X}}_1, \ldots, \bar{\mathcal{X}}_n), \cdot\cdot) = \bar{w}^j(\bar{x}_1, \ldots, \bar{x}_n) \qquad (j = 1, \ldots, m).$$

One can also obtain the transformation of the derivatives

$$\bar{w}_I^j = \frac{\partial \bar{w}^j}{\partial \bar{x}_I} = W_I^j(\cdot\cdot, x_{i'}, w_{I'}^{j'}, \cdot\cdot) \qquad \text{(all } j \text{ and } I) \tag{3}$$

which complete the equations (2). They satisfy the recurrence

$$\sum W_{Ii}^j D_i X_{i'} = D_{i'} W_I^j \qquad \text{(all } i', j, I) \tag{4}$$

and we altogether speak of a *morphism* (2), (3). If there exists the inverse morphism

$$x_i = \bar{X}_i(\cdot\cdot, \bar{x}_{i'}, \bar{w}_{I'}^{j'}, \cdot\cdot), w_I^j = \bar{W}_I^j(\cdot\cdot, \bar{x}_{i'}, \bar{w}_{I'}^{j'}, \cdot\cdot), \tag{5}$$

we speak of an *automorphism* (or: *symmetry*). The structure of the totality of all symmetries unexpectedly manifests as an unheard–of mystery [1],[2]. Just the symmetries are important since they produce the higher–order equivalences of differential equations.

1.2   Example: the Wave Construction

Assuming $n = 1$, we abbreviate

$$x = x_1, \bar{x} = \bar{x}_1, w_r^j = w_{1\cdots1}^j, \bar{w}_r^j = \bar{w}_{1\cdots1}^j \qquad (r \text{ terms}).$$

Let $V = V(x, w_0^1, \ldots, w_0^m, \bar{x}, \bar{w}_0^1, \ldots, \bar{w}_0^m)$ be a given function.

Proposition [3],[4]. *If the implicit system*

$$V = DV = \cdots = D^m V = 0 \qquad (D = \frac{\partial}{\partial x} + \sum w_{r+1}^j \frac{\partial}{\partial w_r^j})$$

admits a solution

$$\bar{x} = X(x, w_0^1, \ldots, w_m^m), \ \bar{w}_0^j = w_0^j(x, w_0^1, \ldots, w_m^m) \quad (j = 1, \ldots, m) \tag{6}$$

such that $DX \neq 0$ and the implicit system

$$V = \bar{D}V = \cdots = \bar{D}^m V = 0 \qquad (\bar{D} = \frac{\partial}{\partial \bar{x}} + \sum \bar{w}_{r+1}^j \frac{\partial}{\partial \bar{w}_r^j})$$

admits a solution

$$x = \bar{X}(\bar{x}, \bar{w}_0^1, \ldots, \bar{w}_m^m), \ w_0^j = \bar{w}_0^j(\bar{x}, \bar{w}_0^1, \ldots, \bar{w}_m^m) \quad (j = 1, \ldots, m) \tag{7}$$

such that $\bar{D}\bar{X} \neq 0$ then (6) and (7) are symmetric inverse of each other.

The Proposition can be extensively generalized for the case $n > 1$ and all the classical Lie contact transformations are involved if $m = 1$. Except for this Lie's favourable subcase, all such symmetries destroy the finite–order jet spaces.

1.3 The Infinitesimal Modification [5],[6],[7]

The ancient "linear approximation" of equations (2) and (3) reads

$$\bar{x}_i = x_i + \varepsilon z_i(\cdot\cdot, x_{i'}, w_{I'}^{j'}, \cdot\cdot), \ \bar{w}_I^j = w_I^j + \varepsilon z_I^j(\cdot\cdot, x_{i'}, w_{I'}^{j'}, \cdot\cdot)$$

where ε is the famed "small parameter". In the rigorous theory, let us instead introduce the vector field

$$Z = \sum z_i \frac{\partial}{\partial x_i} + \sum z_I^j \frac{\partial}{\partial w_I^j} \qquad (z_{Ii}^j = D_i z_I^j - \sum w_{Ii'}^j D_{i'} z_i) \tag{8}$$

with the recurrence following from (4).

   Warning. In contrast to actual convention, we speak of a *variation* and the common term the *generalized* (or: *higher–order*, or: *Lie–Bäcklund*) *infinitesimal transformation* (briefly: the *symmetry*) is retained only for the case when the vector field Z generates a true (local) Lie group of transformations

$$\bar{x}_i[t] = X_i(\cdot\cdot, x_{i'}, w_{i'}^{j'}, \cdot\cdot; t), \ w_I^j[t] = W_I^j(\cdot\cdot, x_{i'}, w_{i'}^{j'}, \cdot\cdot; t) \qquad (-\varepsilon < t < \varepsilon)$$

depending on the group parameter t, that is, for the favourable case when the Lie system

$$\frac{\partial X_i}{\partial t} = z_i(\cdot\cdot, X_{i'}, W_{i'}^{j'}, \cdot\cdot), \ \frac{\partial W_I^j}{\partial t} = z_I^j(\cdot\cdot, X_{i'}, W_{i'}^{j'}, \cdot\cdot) \qquad (X_i|_{t=0} = x_i, W_I^j|_{t=0} = w_I^j)$$

admits a solution. While all variations are given by simple explicit formula (8), the totality of all infinitesimal symmetries is unknown.

1.4 Example: Variations and Symmetries

We again suppose $n = 1$. The vector field

$$D = Z = \frac{\partial}{\partial x} + \sum w_{r+1}^j \frac{\partial}{\partial w_r^j} \qquad (z_1 = 1, z_{1\cdots 1}^j = z_r^j = w_{r+1}^j \text{ with } r \text{ terms})$$

is clearly a variation but not a symmetry since the Lie system

$$\frac{\partial X}{\partial t} = z_1 = 1, \ \frac{\partial W_r^j}{\partial t} = W_{r+1}^j = DW_r^j \qquad (X|_{t=0} = x, W_r^j|_{t=0} = w_r^j)$$

for the corresponding Lie group is contradictory. On the contrary, the vector field

$$Z = \sum w_{r+1}^1 \frac{\partial}{\partial w_r^k} \qquad (m \geq 2; \text{the sum over } k = 2, \ldots, m \text{ and } r = 0, 1, \ldots)$$

generates the very simple "higher–order Lie group" of the morphisms

$$\bar{x}[t] = x, \bar{w}_r^1[t] = w_r^1, \bar{w}_r^k[t] = w_r^k + tw_{r+1}^1 \quad (k = 2, \ldots, m; r = 0, 1, \ldots).$$

This very simple group does not preserve many of the classical concepts, even the order of the differential equations.

1.5 External Differential Equations [7]

Let us denote $\mathbf{M}(m, n)$ the space with coordinates (1) from now on. (This is the classical *jet space* without any differential constraints.) Differential equations are traditionally interpreted as the subspace $\mathbf{M} \subset \mathbf{M}(m, n)$ defined by certain equations

$$D_{i_1} \cdots D_{i_r} f^k = 0 \qquad (k = 1, \ldots, K; i_1, \ldots, i_r = 1, \ldots, n; r = 0, 1, \ldots) \tag{9}$$

where $f^1, \ldots, f^K$ are given functions of variables (1). Together with the "primary" equations $f^1 = \cdots = f^K = 0$, also all consequences (9) are taken into account. The compatibility is implicitly contained in the definition.

Let us recall the morphisms. By using the more precise pull–back notation

$$\mathbf{m}^* x_i = \bar{x}_i, \mathbf{m}^* w_I^j = \bar{w}_I^j$$

in the equations (2) and (3), they may be interpreted as a mapping $\mathbf{m} : \mathbf{M}(m,n) \to \mathbf{M}(m,n)$, the *morphism of the space* $\mathbf{M}(m,n)$. Assuming moreover $\mathbf{mM} \subset \mathbf{M}$, then $\mathbf{m}$ is said to be the *external morphism* of $\mathbf{M}$ (or: of differential equations (9)) and the *external automorphism* (or: *external symmetry*) in the invertible case (5).

The infinitesimal concepts are analogous: if the vector field (8) is tangent to $\mathbf{M}$ (and therefore Z is a vector field on $\mathbf{M}$ as well) then Z is called the *external variation* of $\mathbf{M}$ (or: of differential equations (9)) and if Z moreover generates a group, we have the *external infinitesimal symmetry*.

1.6   Internal Differential Equations [7]

The external morphism of $\mathbf{M}$ whose restriction to $\mathbf{M}$ is invertible is the *internal* symmetry. The external variation Z of $\mathbf{M}$ which generates a Lie group on $\mathbf{M}$ is the *internal* infinitesimal symmetry. These internal concepts are in fact independent of the localization $\mathbf{M}$ in the ambient space $\mathbf{M}(m,n)$. The reasons are as follows.

Let $\Omega(m,n)$ be the module of all *contact forms*

$$\omega = \sum f_I^j \omega_I^j \qquad (\text{finite sum}, \omega_I^j = \mathrm{d}w_I^j - \sum w_{Ii}^j \mathrm{d}x_i) \tag{10}$$

on the space $\mathbf{M}(m,n)$. One can observe that the recurrence (3) is equivalent to the inclusion $\mathbf{m}^* \Omega(m,n) \subset \Omega(m,n)$. Analogously the recurrence in (8) is expressed by the inclusion $\mathcal{L}_Z \Omega(m,n) \subset \Omega(m,n)$ for the Lie derivative.

Let Ω be the restriction of the module $\Omega(m,n)$ to the subspace $\mathbf{M} \subset \mathbf{M}(m,n)$. The morphism $\mathbf{m}$ restricted to $\mathbf{M}$ clearly satisfies $\mathbf{m}^* \Omega \subset \Omega$. Analogously the variation Z restricted to $\mathbf{M}$ satisfies the inclusion $\mathcal{L}_Z \Omega \subset \Omega$. Therefore both concepts are characterized without the use of the ambient space $\mathbf{M}(m,n)$. However more is true: *even the module Ω itself* can be characterized in abstract terms, we speak of a *diffiety Ω* on $\mathbf{M}$, see below.

Altogether we obtain the *internal* theory for the differential equations (9) not affected by the inclusion into the ambient jet space $\mathbf{M}(m,n)$.

1.7   Example: the Internal Symmetry

Assuming $n = 1$, we introduce the subspace $\mathbf{M} \subset \mathbf{M}(m,1)$ defined by the equations

$$D^r w_2^j = w_{r+2}^j = 0 \qquad (j = 1, \ldots, m; \, r = 0, 1, \ldots).$$

Then the morphism $\mathbf{m} : \mathbf{M}(m,1) \to \mathbf{M}(m,1)$ where

$$\mathbf{m}^* x = \bar{x} = x, \mathbf{m}^* w_r^j = \bar{w}_r^j = w_r^j + w_{r+1}^j \quad (j = 1, \ldots, m; \, r = 0, 1, \ldots)$$

is clearly noninvertible but the restriction to $\mathbf{M}$

$$\mathbf{m}^* x = x, \mathbf{m}^* w_0^j = w_0^j + w_1^j, \mathbf{m}^* w_1^j = w_1^j \quad (j = 1, \ldots, m)$$

is a symmetry. Analogously the vector field $Z = D$ is a mere variation on $\mathbf{M}(m,1)$ but generates the Lie group

$$\bar{x}[t] = x, \bar{w}_0^j[t] = w_0^j + t w_1^j, \bar{w}_1^j[t] = w_1^j \quad (j = 1, \ldots, m)$$

on the space $\mathbf{M}$. So we have the internal but not the external symmetries.

1.8 Use of the Pfaffian Equations

For instance, let us mention the equation

$$\mathrm{d}z = p\mathrm{d}x + F(x,y,z,p)\mathrm{d}y$$

where x, y, z, p may be regarded as the coordinates of the underlying space. The solutions $z = z(x,y)$ parametrized with x, y satisfy

$$\frac{\partial z}{\partial y} = F(x,y,z,\frac{\partial z}{\partial x}).$$

However *the same* Pfaffian equation

$$\mathrm{d}\bar{z} = -x\mathrm{d}p + F(x,y,\bar{z}+px,p)\mathrm{d}y \quad (\bar{z} = z - px)$$

admits the solutions $\bar{z} = \bar{z}(p,y)$ which satisfy

$$\frac{\partial \bar{z}}{\partial y} = F(-\frac{\partial \bar{z}}{\partial p},y,\bar{z}-p\frac{\partial \bar{z}}{\partial p},p).$$

And quite analogously, *the same* Pfaffian equation

$$\mathrm{d}y = \frac{1}{F}\mathrm{d}z - \frac{p}{F}\mathrm{d}x \quad (F \neq 0, F_p \neq 0)$$

admits the solutions $y = y(z,x)$ which satisfy

$$\frac{\partial y}{\partial x} = -p\frac{\partial y}{\partial z} \quad (F(x,y,z,p) = 1/\frac{\partial y}{\partial z} \text{ determines } p = p(x,y,z,\frac{\partial y}{\partial z})).$$

We conclude. A Pfaffian equation represents many formally quite dissimilar but in fact equivalent differential equations according to the additional choice of the dependent and the independent variables. It follows that a coordinate–free theory should be expressed in terms of the Pfaffian equations.

1.9 Towards the Diffieties

Let us finally recall the subspace $\mathbf{i} : \mathbf{M} \subset \mathbf{M}(m,n)$ in the more precise pull–back notation. Then the module $\Omega = \mathbf{i}^*\Omega(m,n)$ is interpreted as the restriction of the module of all contact forms to the subspace $\mathbf{M}$ of the jet space $\mathbf{M}(m,n)$. The definition equations (9) imply that vector fields $D_1, \ldots, D_n$ are tangent to the subspace $\mathbf{M}$ and therefore may be regarded as vector fields on $\mathbf{M}$ as well. They satisfy the crucial identity

$$\mathcal{L}_{D_{i'}}\omega_I^j = \mathcal{L}_{D_{i'}}(\mathrm{d}w_I^j - \sum w_{Ii}^j\mathrm{d}x_i) = \mathrm{d}w_{Ii'}^j - \sum w_{Iii'}^j\mathrm{d}x_i = \omega_{Ii'}^j$$

which is clearly true even for the restrictions $\mathbf{i}^*\omega_I^j$ of the forms ω_I^j to the space $\mathbf{M}$. Let $\Omega(m,n)_l \subset \Omega(m,n)$ be the submodule of all contact forms of the order l at most, hence $|I| = r \leq l$ in all summands (9). Obviously

$$\Omega(m,n)_l + \sum \mathcal{L}_{D_{i'}}\Omega(m,n)_l = \Omega(m,n)_{l+1} \qquad (\text{all } l) \tag{11}$$

and this property is true even for the restrictions to $\mathbf{M}$, i.e., for the submodules

$$\Omega_l = \mathbf{i}^*\Omega(m,n)_l \subset \mathbf{i}^*\Omega(m,n) = \Omega. \tag{12}$$

We have in fact discovered the crucial property of the diffieties Ω, see below.

We conclude. Differential equations should be represented by the Pfaffian system $\omega = 0$ ($\omega \in \Omega$) on the space $\mathbf{M}$ where the modules Ω are described in abstract terms (the *internal theory*) by using only the property (12). Then the actual choice of the dependent and the independent variables is irrelevant (the *absolute approach*) in the sense that the quite general formulae (3) are admitted.

2   Fundamental Concepts

We deal with smooth and local category of manifolds and mappings. Our notational convention for a mapping $\mathbf{m} : \mathbf{M} \to \bar{\mathbf{M}}$ of manifolds allows the definition domain to be a proper open subset of $\mathbf{M}$. Then the parade of the primary concepts denoted by (I)–(VI) looks as follows.

(I) On the manifolds [5]

Besides the occasional use of the common finite–dimensional spaces, we mainly deal with manifolds $\mathbf{M}$ modelled on $\mathbb{R}^\infty$, i.e., there are coordinates $h^j : \mathbf{M} \to \mathbb{R}$ ($j = 1, 2, \ldots$) such that the ring $\mathcal{F}$ ($= \mathcal{F}(\mathbf{M})$, the abbreviation whenever possible) of admissible functions $f : \mathbf{M} \to \mathbb{R}$ involves just the (smooth) composite functions $f = F(h^1, \ldots, h^{m(f)})$. Then the $\mathcal{F}$–module Φ ($= \Phi(\mathbf{M})$) of differential forms $\varphi = \sum f^j \mathrm{d}g^j$ (finite sum with $f^j, g^j \in \mathcal{F}$) and the $\mathcal{F}$–module $\mathcal{T}$ ($= \mathcal{T}(\mathbf{M})$) of vector fields Z on the space $\mathbf{M}$ make a good sense. It should be noted that the vector fields are regarded as $\mathcal{F}$–linear functions $Z : \Phi \to \mathcal{F}$ where

$$\mathrm{d}f(Z) = Zf, \ \varphi(Z) = Z \rfloor \varphi = \sum f^j Z g^j$$

and if $\varphi^1, \varphi^2, \ldots$ is a basis of module Φ, we denote

$$Z = \sum z^j \frac{\partial}{\partial \varphi^j} \quad \text{(infinite series, arbitrary } z^j = \varphi^j(Z) \in \mathcal{F})$$

with the common abbreviation $\partial/\partial f = \partial/\partial \mathrm{d}f$. The familiar rules like

$$\mathcal{L}_Z f = Zf, \ \mathcal{L}_Z \varphi = Z \rfloor d\varphi + d\varphi(Z), \ \mathcal{L}_Z X = [Z, X], \ \mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$$

for the Lie derivative and the Lie bracket do not need any comment.

Let $\mathbf{n} : \mathbf{N} \to \mathbf{M}$ be a mapping of manifolds. If an appropriate part of the family of functions $\mathbf{n}^* h^1, \mathbf{n}^* h^2, \ldots$ can be taken for the coordinates on $\mathbf{N}$, then $\mathbf{n}$ is called an *inclusion* of the *submanifold* $\mathbf{N}$ into the space $\mathbf{M}$. (Since $\mathbf{n}$ is injective, we occasionally identify $\mathbf{N} = \mathbf{n}\mathbf{N} \subset \mathbf{M}$ with the subset of $\mathbf{M}$.) Analogously $\mathbf{n}$ is called a *projection* of $\mathbf{N}$ on the *factorspace* $\mathbf{M}$ if the family $\mathbf{n}^* h^1, \mathbf{n}^* h^2, \ldots$ can be completed by some additional functions to the coordinates on $\mathbf{N}$. (Since $\mathbf{n}^*$ is injective, we occasionally abbreviate $f = \mathbf{n}^* f, \varphi = \mathbf{n}^* \varphi$.) In order to delete singularities, we tacitly suppose the existence of $\mathcal{F}$–bases in all $\mathcal{F}$–modules to appear (unless otherwise stated). We moreover admit the localization at the points $\mathbf{P} \in \mathbf{M}$ in the sense that the $\mathcal{F}$–basis of $\mathcal{F}$–module turns into the $\mathbb{R}$–basis of $\mathbb{R}$–module after the evaluation $\mathbf{P}$ [8]. Alternatively: certain algebraical arrangements at a given point $\mathbf{P} \in \mathbf{M}$ are tacitly applied in a neighbourhood of $\mathbf{P}$ as well and this is possible on open subsets of "generic points" of $\mathbf{M}$.

(II) On the diffieties [5],[8]

For every submodule $\Omega \subset \Phi$, let $\mathcal{H}(\Omega) \subset \mathcal{T}$ be the submodule of all vector fields Z such that $\Omega(Z) = 0$. A submodule $\Omega \subset \Phi$ is called *flat* (or: *satisfying the Frobenius condition*) if any of the equivalent requirements

$$\mathrm{d}\Omega \cong 0 \ (\mathrm{mod} \ \Omega), \ \mathcal{L}_\mathcal{H} \Omega \subset \Omega, \ [\mathcal{H}, \mathcal{H}] \subset \mathcal{H} \qquad (\mathcal{H} = \mathcal{H}(\Omega)) \tag{13}$$

is satisfied. The finite–dimensional flat submodules $\Omega \subset \Phi$ are simple: the Frobenius theorem can be applied and they admit a basis consisting of total differentials. We are however interested just in the infinite–dimensional case.

Definition 2.1. A finite–codimensional submodule $\Omega \subset \Phi$ is called a *diffiety* if there exists a *good filtration* $\Omega_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \cup \Omega_l$ with the finite–dimensional submodules $\Omega_l \subset \Omega$ ($l = 0, 1, \ldots$) such that

$$\mathcal{L}_\mathcal{H} \Omega_l \subset \Omega_{l+1} \quad \text{(all } l), \quad \Omega_l + \mathcal{L}_\mathcal{H} \Omega_l = \Omega_{l+1} \quad (l \text{ large enough}). \tag{14}$$

A *pre–diffiety* need not satisfy the codimensionality requirement.

We deal only with the diffieties unless otherwise stated. The pre–diffieties will appear later on and will be reduced to the common diffieties.

Definition 2.2. Denoting $n \; (= n(\Omega)) = \dim \Phi/\Omega = \dim \mathcal{H}$, functions $x_1, \ldots, x_n$ are called *independent variables* of diffiety Ω if the differentials $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ are linearly independent mod Ω. Then the *total derivatives* $D_1, \ldots, D_n \in \mathcal{H}$ defined by

$$D_i x_i = 1, \; D_{i'} x_i = 0, \; \Omega(D_i) = 0 \qquad (i, i' = 1, \ldots, n; \; i \neq i')$$

constitute a basis of module $\mathcal{H}$ and the *contact forms*

$$\omega\{f\} = \mathrm{d}f - \sum D_i f \mathrm{d}x_i \quad (f \in \mathcal{F})$$

generate the diffiety Ω.

The link to the classical approach can be succinctly described as follows.

Let us consider differential equations $\mathbf{i} : \mathbf{M} \subset \mathbf{M}(m, n)$ in the sense (9). It was already clarified in Preface 1.9 that the restriction $\Omega = \mathbf{i}^*\Omega(m, n)$ of the module $\Omega(m, n)$ to the space $\mathbf{M}$ is a diffiety with the terms $\Omega_l = \mathbf{i}^*\Omega(m, n)_l$ of the good filtration. Let us conversely start with a diffiety Ω. Due to (14), there exist generators

$$\mathcal{L}_{D_{i_1}} \cdots \mathcal{L}_{D_{i_r}} \omega^k \quad (k = 1, \ldots, K; \; i_1, \ldots, i_r = 1, \ldots, n; \; r = 0, 1, \ldots)$$

of module Ω. Denoting $\omega^k = \sum a_j^k \mathrm{d}h^j$, the Pfaffian system $\omega = 0 \; (\omega \in \Omega)$ clearly reads

$$\sum f_j^k \frac{\partial h^j}{\partial x_i} = 0, \quad D_{i_1} \sum f_j^k \frac{\partial h^j}{\partial x_i} = 0, \quad D_{i_1} D_{i_2} \sum f_j^k \frac{\partial h^j}{\partial x_i} = 0, \; \ldots$$

which is a classical (infinitely prolonged) system of differential equations for a *finite number* of functions $h^j = h^j(x_1, \ldots, x_n)$ occuring in the forms $\omega^1, \ldots, \omega^K$.

We conclude. A diffiety supplied with a fixed choice of the dependent and the independent variables is the same as an infinitely prolonged system of differential equations.

(III) On the commutative algebra [5],[7],[8],[9],[10]

Conditions (14) simplify if the original filtration Ω_* is replaced with the gradation

$$\mathrm{Grad}\, \Omega_* = \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \cdots \qquad (\mathcal{M}_l = \Omega_l/\Omega_{l-1}, \; \Omega_{-1} = 0).$$

Then the Lie derivative

$$\mathcal{L}_D : \Omega_l \to \Omega_{l+1} \qquad (D \in \mathcal{H}(\Omega))$$

turns into the $\mathcal{F}$–linear mapping $D : \mathcal{M} \to \mathcal{M}$ such that

$$D[\omega] = [\mathcal{L}_D \omega] \in \mathcal{M}_{l+1} \qquad (\omega \in \Omega_l, [\omega] \in \mathcal{M}_l, D \in \mathcal{H})$$

where the square brackets denote the factorization. But more is true. Let

$$\mathcal{A} \; (= \mathcal{A}(\mathcal{H})) = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \cdots \qquad (\mathcal{A}_0 = \mathcal{F}, A_1 = \mathcal{H}, \mathcal{A}_2 = \mathcal{H} \odot \mathcal{H}, \ldots)$$

be the algebra of homogeneous polynomials over $\mathcal{H}$. We obtain even the $\mathcal{A}$–module structure on $\mathcal{M}$. For instance

$$D_{i_1} \cdots D_{i_r}[\omega] = [\mathcal{L}_{D_{i_1}} \cdots \mathcal{L}_{D_{i_r}} \omega] \in \mathcal{M}_{l+r} \qquad (\omega \in \Omega_r, D_{i_1} \cdots D_{i_r} \in \mathcal{A}_r),$$

$$D_{i_1} \cdots D_{i_r}[\omega\{f\}] = [\omega\{D_{i_1} \cdots D_{i_r} f\}].$$

Warning. The algebraical calculations with the $\mathcal{F}$–module are performed at a fixed point of $\mathbf{M}$. So we deal with "smooth families" of $\mathbb{R}$–modules. It follows that the classical algebra can be applied.

In particular we recall the *Hilbert polynomial*

$$\dim \mathcal{M}_l = e_\nu \binom{l}{\nu} + \cdots + e_0 \binom{l}{0} \qquad (l \text{ large enough}, e_\nu \neq 0)$$

with integer coefficients, alternatively

$$\dim \Omega_l = e_\nu \binom{l+1}{\nu+1} + \cdots + e_{-1} \binom{l+1}{0} \qquad (l \text{ large enough}, e_\nu \neq 0).$$

Then the degree $\nu \ (= \nu(\Omega))$ and the integer $\mu \ (= \mu(\Omega)) = e_\nu > 0$ do not depend on the choice of the filtration Ω_*.

Definition 2.3. We claim that *the solution* of diffiety Ω *depends* on $\mu(\Omega)$ functions of $\nu(\Omega)+1$ variables. Diffiety Ω is *overdetermined, determined, or underdetermined* according to whether $\nu + 1 < n - 1, \nu + 1 = n - 1$, or $\nu + 1 > n - 1$, respectively.

One can observe that we have intentionally not yet introduced the concept of a solution of diffiety Ω. Definition 2.3 therefore designates a more *formal property* of diffiety Ω, however, it is in full accordance with quite opposite approach [11],[12],[13],[14],[15],[16] in the theory of exterior differential systems.

Remark. The above reasonings make sense even for the much easier case of the finite–dimensional underlying space $\mathbf{M}$ which need not be separately discussed here. Let us only recall the familiar Frobenius theorem which ensures that then the diffiety $\Omega \subset \Phi(\mathbf{M})$ has a basis $\mathrm{d}f^1, \ldots, \mathrm{d}f^\mu$ and the solution depends on $\mu = \dim \Omega = e_{-1}$ parameters. We formally put $\nu \ (= \nu(\Omega)) = -1$.

(IV) On the symmetries [5],[7]

Admissible mappings $\mathbf{m} : \mathbf{M} \to \bar{\mathbf{M}}$ between manifolds satisfy $\mathbf{m}^* \mathcal{F}(\bar{\mathbf{M}}) \subset \mathcal{F}(\mathbf{M})$ whence $\mathbf{m}^* \Phi(\bar{\mathbf{M}}) \subset \Phi(\mathbf{M})$.

Definition 2.4. Let $\Omega \subset \Phi(\mathbf{M})$ and $\bar{\Omega} \subset \Phi(\bar{\mathbf{M}})$ be diffieties. Then the mapping $\mathbf{m} : \mathbf{M} \to \bar{\mathbf{M}}$ is said to be a *morphism of diffieties* if $\mathbf{m}^* \bar{\Omega} \subset \Omega$. Invertible morphisms with $n(\Omega) = \bar{n}(\Omega)$ are *isomorphisms* and then $\mathbf{m}^* \bar{\Omega} = \Omega$. Assuming $\mathbf{M} = \bar{\mathbf{M}}$ and $\Omega = \bar{\Omega}$, invertible morphisms are called *symmetries* (or: *automorphisms*).

Three subcases of symmetries can be informally mentioned as follows. First, if a *given* filtration is preserved (Figure 1a). Second, if an *unknown* filtration is preserved (Figure 1b). Third, if no *finite–dimensional subspace* of Ω is preserved (Figure 1c).
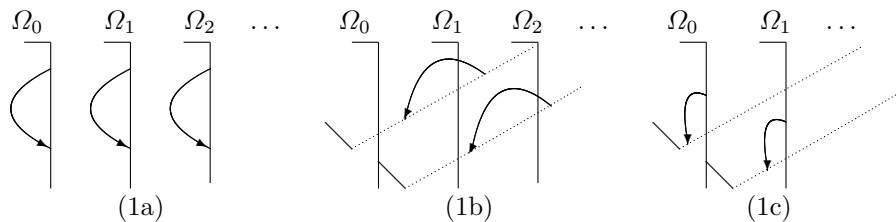

Figure 1: Subcases of symmetries.

The common methods of the general equivalence [16] can be directly applied only to the subcase (1a) and with slight adaptations even to the subcase (1b) not occuring in the classical theory. Alas, the common methods fail in the subcase (1c). No universal finite algorithm for the determination of *all* symmetries or equivalences of diffieties is known.

(V) On the variations [5],[7]

We expect that the determination of "approximative symmetries" is easier. They are realized by vector fields.

Definition 2.5. *Variations* $Z \in \mathcal{T}(\mathbf{M})$ of a diffiety $\Omega \subset \Phi(\mathbf{M})$ are defined by the condition $\mathcal{L}_Z \Omega \subset \Omega$. Variations generating a Lie group are called *infinitesimal symmetries.*

The Figure 1 with arrows denoting the Lie derivative $\mathcal{L}_Z$ can be mentioned as well, however, the comments are different compared to the previous case of the symmetries **m**. In more detail, the subcases (1a) and (1b) concern the variations Z which generate a group, that is, we have the infinitesimal symmetries. Then, paradoxically, the "true variations" of the subcase (1c) do not cause much difficulties.

Lemma 2.1. *A variation* Z *is characterized by the property*

$$(\mathcal{L}_D \omega)(Z) = D\omega(Z) \qquad (D \in \mathcal{H}, \omega \in \Omega). \tag{15}$$

Proof. Inclusion $\mathcal{L}_Z \Omega \subset \Omega$ is equivalent to the identity $\mathcal{L}_Z \mathcal{H} = [Z, \mathcal{H}] \subset \mathcal{H}$ which follows from the equations

$$0 = \Omega(\mathcal{H}), \ 0 = Z(\Omega(\mathcal{H})) = \mathcal{L}_Z \Omega(\mathcal{H}) + \Omega([Z, \mathcal{H}]).$$

So we have

$$D(\omega(Z)) = \mathcal{L}_D \omega(Z) + \omega([D, Z]) = \mathcal{L}_D \omega(Z).$$

$\square$

This simple Lemma 2.1 involves the common *linearization procedure* [12],[13] as a particular subcase when $\omega = \omega\{f\}$. If a *variation* Z is represented by the series (13) then the coefficients $\varphi^j(Z)$ with appropriately chosen forms $\varphi^j \in \Omega$ can be effectively described. On the contrary, the study of the *infinitesimal symmetries* Z is rather difficult: they satisfy one additional condition, the invariance of some filtration. In general, there are too many filtrations and this prevents us from resolving the symmetry problem completely.

(VI) On the evolutional diffieties

The infinitesimal symmetry $Z \in \mathcal{T}(\mathbf{M})$ of a diffiety $\Omega \subset \Phi(\mathbf{M})$ is a classical concept with simple geometrical significance, the flow on the underlying space $\mathbf{M}$. The variations Z look rather ambiguous in this respect, they are rather arguably identified with virtual flows on the vague space of solutions of diffiety Ω [12],[13]. A rigorous view is however possible [5].

Let us introduce the direct product $\bar{\mathbf{M}} = \mathbf{M} \times \mathbb{R}$ of manifolds with coordinate t in the factor $\mathbb{R}$. Omitting the technicalities, a function $f \in \Phi(\mathbf{M})$ can be regarded as a function on $\bar{\mathbf{M}}$ (independent of t) and analogously for the forms $\varphi \in \Phi(\mathbf{M})$. If $\varphi^1, \varphi^2, \ldots$ is a basis of module $\Phi(\mathbf{M})$ then $dt, \varphi^1, \varphi^2, \ldots$ constitute a basis of $\Phi(\bar{\mathbf{M}})$. The vector fields $\bar{Z} \in \mathcal{T}(\bar{\mathbf{M}})$ can be described as follows. There are *horizontal* vector fields $H \in \mathcal{T}(\bar{\mathbf{M}})$ satisfying $Ht = 0$ and they may be identified with vector fields $H(t) \in \mathcal{T}(\mathbf{M})$ depending on parameter t which are distributed over $\bar{\mathbf{M}}$ by the t–shifts along the $\mathbb{R}$–component. Then, by using the obvious *vertical* vector field $\partial/\partial t$, we obtain the unique decomposition

$$\bar{Z} = H + \bar{f}\frac{\partial}{\partial t} \quad (\bar{f} = \bar{Z}t, Ht = 0, Hf = H(t)f \text{ if } f \in \mathcal{F}(\mathbf{M}) \subset \mathcal{F}(\bar{\mathbf{M}}))$$

into the horizontal and the vertical summands.

With this preparation, the following diffieties of rather special kind provide the rigorous geometrical sense of the variations.

Definition 2.6. Let $Z(t) \in \mathcal{T}(\mathbf{M})$ be a variation depending on parameter t of a diffiety $\Omega \subset \Phi(\mathbf{M})$. We introduce the *evolutional diffiety* $\bar{\Omega} \subset \Phi(\mathbf{M} \times \mathbb{R})$ with generators

$$\omega(Z(t))dt - \omega \in \bar{\Omega} \subset \Phi(\bar{\mathbf{M}}) \qquad (\omega \in \Omega). \tag{16}$$

Alternatively, the module $\mathcal{H}(\bar{\Omega})$ is generated by all vector fields

$$E = Z(t) + \frac{\partial}{\partial t} \in \mathcal{H}(\bar{\Omega}) \subset \mathcal{T}(\bar{\mathbf{M}}) \text{ and } D \in \mathcal{H}(\Omega) \tag{17}$$

where the vector fields $D \in \mathcal{H}(\Omega)$ are identified with horizontal vector fields distributed over $\mathbf{M} \times \mathbb{R}$.

Definition 2.7. An inclusion $\mathbf{n} : \mathbf{N} \to \mathbf{M}$ ($\mathbf{N} \subset \mathbb{R}^n, n = n(\Omega)$) is said to be a *solution* of diffiety $\Omega \subset \Phi(\mathbf{M})$ if $\mathbf{n}^* \Omega = 0$.

In accordance with tradition, we identify $\mathbf{N} = \mathbf{nN}$ which better corresponds to the intuition: the diffiety Ω identically vanishes on the subspace $\mathbf{N} = \mathbf{nN} \subset \mathbf{M}$ of the total space or, equivalently, all vector fields $D \in \mathcal{H}(\Omega)$ are tangent to this subspace $\mathbf{N} = \mathbf{nN} \subset \mathbf{M}$ of the dimension $n = n(\Omega)$. We recall that the existence of solutions is a highly delicate task [11],[17],[15], to put it mildly. (Informally, this is in fact a mere rigid formalism for the familiar concept: if $x_1, \ldots, x_n$ are independent variables of diffiety $\Omega \subset \Phi(\mathbf{M})$ and $h^1, h^2, \ldots$ are coordinates on $\mathbf{M}$ then the solution is given by such formulae $h^j = h^j(x_1, \ldots, x_n)$ that satisfy the Pfaffian system $\omega = 0$ ($\omega \in \Omega$).)

The resulting point is as follows. Let $\bar{\mathbf{N}} \subset \bar{\mathbf{M}}$ be a solution of the evolutional diffiety $\bar{\Omega}$, hence $\dim \bar{\mathbf{N}} = \dim \mathcal{H}(\bar{\Omega})$ and vector fields $\bar{D} \in \mathcal{H}(\bar{\Omega})$ are tangent to $\bar{\mathbf{N}}$. Let $\mathbf{N} \subset \mathbf{M}$ be the projection of $\bar{\mathbf{N}}$. Since the vector fields D distributed over $\bar{\mathbf{M}}$ are tangent to $\bar{\mathbf{N}}$, we conclude that the projections of the level subsets $t = const.$ of $\bar{\mathbf{N}}$ on $\mathbf{N}$ are solutions of Ω. On the other hand, E is tangent to $\bar{\mathbf{N}}$ as well and generates a one–parameter group on $\bar{\mathbf{N}}$ where the $\mathbb{R}$–component involves mere translations $t \to t + c$. It follows that the level sets on $\bar{\mathbf{N}}$ are permuted and the projections of the level sets in $\mathbf{N}$, the solutions of Ω, are permuted as well.

We summarize: a variation (possibly depending on a parameter) generates many flows, but each only on a rather narow family of solutions of diffiety Ω.

Proposition 2.1. *Let $\bar{\mathbf{N}} \subset \bar{\mathbf{M}} = \mathbf{M} \times \mathbb{R}$ be a solution of the evolutional diffiety $\bar{\Omega} \subset \Phi(\bar{\mathbf{M}})$ and $\mathbf{N} \subset \mathbf{M}$ the natural projection of $\bar{\mathbf{N}}$. Then the level subsets $t = const.$ of $\bar{\mathbf{N}}$ are projected on the solutions of diffiety $\Omega \subset \Phi(\mathbf{M})$. The vector field $E \in \mathcal{T}(\bar{\mathbf{M}})$ generates a Lie group on $\bar{\mathbf{N}}$ and its projection $Z(t) \in \mathcal{T}(\mathbf{M})$ permutes the solutions of Ω contained in $\mathbf{N}$.*

The multi–parameter evolution diffieties for the case of a *finite–dimensional* Lie algebra of variations Z can be analogously introduced as well.

Remark. A huge literature on the infinitesimal symmetries in finite–order jet spaces exists. These are the *order–preserving* Lie symmetries of Figure 1a. We cannot refer to any literature systematically devoted to the *order–destroying* symmetries of Figure 1b. The main and rather original ideas of monograph [13] concern the abstract theory of variations of Figure 1c, alas, with lack of any convincing examples and applications. The alternative approach [12],[13] to the diffieties is worth mentioning, however, large preparatory parts are devoted to finite–order jet spaces and *neither the results nor the methods of proofs can be carried over the diffieties,* they are even misleading in many respects.

3   The Involutiveness

In the classical external theory, the involutivity ensures that a given finite system of differential equations is compatible. In the classical internal theory, the involutivity ensures the same for a finite Pfaffian system. We are interested in diffieties Ω where the compatibility is already attained. Then the involutivity describes the structure of the higher–order summands of the $\mathcal{A}$–module $\mathcal{M}$ corresponding to a *given good filtration* Ω_* and this is a mere algebra.

Let $Z_1, \ldots, Z_n$ ($n = n(\Omega)$) be a basis of module $\mathcal{H}(= \mathcal{H}(\Omega))$ and $\mathcal{A}(i) \subset \mathcal{A}$ ($i = 0, \ldots, n$) the ideal generated by $Z_1, \ldots, Z_i$. In particular $\mathcal{A}(0) = 0$ and

$$\mathcal{A}(n) = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots = \mathfrak{m} \subset \mathcal{A} \qquad (\mathcal{A}_1 = \mathcal{H}, \mathcal{A}_2 = \mathcal{H} \odot \mathcal{H}, \ldots)$$

is the *maximal ideal*. We introduce the factormodules

$$\mathcal{M}(i) = \mathcal{M}/\mathcal{A}(i)\mathcal{M} = \mathcal{M}(i)_0 \oplus \mathcal{M}(i)_1 \oplus \cdots \quad (\mathcal{M}(i)_l = \mathcal{M}_l/\mathcal{A}(i)\mathcal{M} \cap \mathcal{M}_l)$$

which are $\mathcal{A}$–modules as well. In particular

$$Z_{i+1} : \mathcal{M}(i)_l \to \mathcal{M}(i)_{l+1} \qquad (i = 0, \ldots, n-1). \tag{18}$$

Definition 3.1. The basis $Z_1, \ldots, Z_n$ is called *ordinary* (or: *generic* [5]) for a given good filtration Ω_* if (18) are injections for l large enough.

Theorem 3.1 ([5],[8]). *There exists an ordinary basis of $\mathcal{H}$.*

In full generality, this is rather nontrivial result, however, for all current examples, such a basis can be easily found: the vector fields $Z_1, Z_2, \ldots$ should not be too special, see also below. We are passing to the main topic.

Definition 3.2. The basis $Z_1, \ldots, Z_n$ of $\mathcal{H}$ is called *quasiregular* if (18) are injections for all $l \geq 1$. A filtration Ω_* is called *involutive* if there exists a quasiregular basis and moreover $\mathcal{H}\mathcal{M}_l = \mathcal{M}_{l+1}$ ($l \geq 0$).

The last condition is clearly equivalent to $\Omega_l + \mathcal{L}_{\mathcal{H}}\Omega_l = \Omega_{l+1}$ ($l \geq 0$) which can be ensured by a simple change of the original filtration. Let us introduce the c–lift Ω_{*+c} ($c = 0, 1, \ldots$) of the original filtration Ω_* such that

$$\Omega_{*+c} = \bar{\Omega}_* : \bar{\Omega}_0 = \Omega_c \subset \bar{\Omega}_1 = \Omega_{c+1} \subset \cdots \subset \Omega = \cup\bar{\Omega}_l = \cup\Omega_{l+c}.$$

Due to the second requirement (13), it follows easily that Theorem 3.1 is equivalent to:

Theorem 3.2. *Every lift Ω_{*+c} with c large enough is involutive.*

This provides the essence of all prolongations into the involutivity mechanisms [11],[16],[5],[18]. Also the singular solutions can be included, see below.

We conclude with a brief description of the ordinary basis. The first term Z_1 appears as follows [8]. There is a finite family of prime ideals $\mathfrak{p} \subset \mathcal{A}, \mathfrak{p} \neq \mathcal{A}$, *associated* to the module $\mathcal{M}$, that is, such that there exists a submodule of $\mathcal{M}$ isomorphic to $\mathcal{A}/\mathfrak{p}$. The multiplications $Z : \mathcal{M}_l \to \mathcal{M}_{l+1}$ ($Z \in \mathcal{H}$) are all injective if and only if

$$Z \notin \mathcal{A}_1 \cap (\cup\mathfrak{p}) = \mathcal{H} \cap (\cup\mathfrak{p}). \tag{19}$$

If all $\mathfrak{p} \neq \mathfrak{m}$ then such $Z = Z_1$ does exist. If however $\mathfrak{m}$ belongs to the associated ideals then the submodules of $\mathcal{M}$ isomorphic to $\mathcal{A}/\mathfrak{m} = \mathcal{F}$ can be included into the submodule $\mathcal{M}_0 \subset \mathcal{M}$ if the original filtration Ω_* is replaced with a c–lift large enough. The injectivity $Z = Z_1 : \mathcal{M}_l \to \mathcal{M}_{l+1}$ ($l \geq 1$) is ensured. The following terms $Z_2, \ldots, Z_n$ appear by analogous construction applied to the $\mathcal{A}$–modules $\mathcal{M}(1), \ldots, \mathcal{M}(n-1)$ instead of $\mathcal{M} = \mathcal{M}(0)$.

Altogether we conclude that the terms $Z_1, \ldots, Z_n$ of the ordinary basis should not lie in a finite family of certain linear subspaces of the $\mathcal{F}$–module $\mathcal{H}$. We will succinctly express this property by saying that they are "not too special". It follows that the total derivatives $D_1, \ldots, D_n \in \mathcal{H}$ are "not too special" for an appropriate "not too special" choice of the independent variables $x_1, \ldots, x_n$.

Remark. The primary approach to the compatibility of the systems of differential equations directly use the commutativity $\partial^2/\partial x \partial y = \partial^2/\partial y \partial x$ of various second derivatives which results in perfect ultimate theory [14]. However, though this theory can be effectively applied to particular problems, the calculations strongly depend on subtle formal details. On the contrary, the É. Cartan's involutivity [15] subsequently completed with the prolongation procedure [16] is of the geometrical nature. In the actual rigorous expositions, this topic belongs to the most difficult tasks even though the powerful tools of the commutative and the homological algebra are applied [11],[18]. The classical involutivity concept differs from ours in Definition 3.2, since the involutivity of a finite–order system of differential equations and of a finite Pfaffian system is introduced [11],[17],[15],[16],[5] and Theorem 3.2 declares the involutivity of every Pfaffian system $\omega = 0$ ($\omega \in \Omega_l$, l fixed and large enough) in the common classical sense, that

is, we have involutivity together with the prolongation procedure involved. It should be noted that the involutivity ensures the finite length of algorithms. This concerns the case (1a) of Figure 1 and prevents us from completely discussing the remaining cases (1b) and (1c) in the present time.

We recall that except for some advanced applications in geometry [11],[17],[16], the involutivity can be easily verified in all current examples of the analysis. On the other hand, the following technical tool of standard filtrations directly inspired just by the abstract involutivity seems to be useful even for the solution of various "simple" earthly problems to follow.

4   The Standard Filtrations

Theorems 3.1 and 3.2 concern the higher–order terms $\mathcal{M}[i]_l$ of the $\mathcal{A}$–module $\mathcal{M}[i]$. Returning to the original filtration Ω_*, they describe a certain property of the higher–order terms Ω_l of a good filtration Ω_*.

For given vector fields $Z_1, Z_2, \ldots \in \mathcal{H}$, let us introduce the large series of accompanying "rough" filtrations

$$\Omega(Z_1)_* : \Omega(Z_1)_0 \subset \Omega(Z_1)_1 \subset \ldots \subset \Omega \qquad (\Omega(Z_1)_l = \sum \mathcal{L}_{Z_1}^k \Omega_l),$$
$$\Omega(Z_1, Z_2)_* : \Omega(Z_1, Z_2)_0 \subset \Omega(Z_1, Z_2)_1 \subset \ldots \subset \Omega \ (\Omega(Z_1, Z_2)_l = \sum \mathcal{L}_{Z_2}^k \Omega(Z_1)_l),$$
$$\cdots$$

of diffiety Ω. For every submodule $\Theta \subset \Phi$ and a vector field $Z \in \mathcal{H}(\Theta)$ we moreover introduce the submodule $\mathrm{Ker}_Z \Theta \subset \Theta$ of all $\vartheta \in \Theta$ with $\mathcal{L}_Z \vartheta \in \Theta$. (The latter concept will be applied only in the particular case when $\Theta \subset \Omega$ and $Z \in \mathcal{H} = \mathcal{H}(\Omega) \subset \mathcal{H}(\Theta)$.) One can see that Theorems 3.1 and 3.2 are equivalent to the equalities

$$\begin{aligned} \mathrm{Ker}_{Z_1} \Omega_{l+1} &= \Omega_l, \\ \mathrm{Ker}_{Z_2} \Omega(Z_1)_{l+1} &= \Omega(Z_1)_l, \\ \mathrm{Ker}_{Z_3} \Omega(Z_1, Z_2)_{l+1} &= \Omega(Z_1, Z_2)_l, \\ &\cdots \end{aligned} \qquad (20)$$

valid for l large enough and not too special $Z_1, Z_2, \ldots \in \mathcal{H}$. Our next aim is to ensure (20) for all values of l after appropriate adjustments and this is possible if certain obstructions $\mathcal{R}^0, \mathcal{R}^1, \ldots$ are absent.

Let us start with the *first equality* (20). Abbreviating $X = Z_1$, we may suppose $\mathrm{Ker}_X \Omega_l = \Omega_{l-1} \ (l \geq L)$ and consider the inclusions

$$\cdots \supset \Omega_L = \mathrm{Ker}_X \Omega_{L+1} \supset \Omega_{L-1} = \mathrm{Ker}_X \Omega_L \supset \mathrm{Ker}_X^2 \Omega_L \supset \mathrm{Ker}_X^3 \Omega_L \supset \cdots.$$

The strict inclusions terminate with certain equalities $\mathrm{Ker}_X^k \Omega_L = \mathrm{Ker}_X^{k+1} \Omega_L \ (k \geq K)$ and so we obtain the new filtration

$$\begin{aligned} \bar{\Omega}_* : \bar{\Omega}_0 &= \mathrm{Ker}_X^{K-1} \Omega_L \subset \bar{\Omega}_1 = \mathrm{Ker}_X^{K-2} \Omega_L \subset \cdots \\ &\subset \bar{\Omega}_{K-2} = \mathrm{Ker}_X \Omega_L \subset \bar{\Omega}_{K-1} = \Omega_L \subset \bar{\Omega}_K = \Omega_{L+1} \subset \cdots \end{aligned} \qquad (21)$$

of diffiety Ω with strict inclusions together with the submodule

$$\mathcal{R}^0 = \mathrm{Ker}_X^K \Omega_L = \mathrm{Ker}_X \bar{\Omega}_0 \subset \bar{\Omega}_0 \subset \Omega.$$

Theorem 4.1 ([5],[8],[9]). *Filtration* (21) *does not depend on the choice of X. The module $\mathcal{R}^0$ is flat and does not depend even on the choice of the original good filtration Ω_*.*

So we may denote $\mathcal{R}^0 = \mathcal{R}^0(\Omega)$ and there are equalities

$$\mathrm{Ker}_X \bar{\Omega}_{l+1} = \bar{\Omega}_l \ (l > 0), \mathrm{Ker}_X \bar{\Omega}_0 = \mathcal{R}^0, \mathrm{Ker}_X \mathcal{R}^0 = \mathcal{R}^0$$

corresponding to the injections

$$X : \bar{\mathcal{M}}_l \to \bar{\mathcal{M}}_{l+1} \ (l > 0), \ \bar{\mathcal{M}}_0/\mathcal{R}^0 \to \bar{\mathcal{M}}_1 \ (\bar{\mathcal{M}}_l = \bar{\Omega}_l/\bar{\Omega}_{l-1})$$

for all not too special vector fields $X \in \mathcal{H}$. The first equality (20) is universal if and only if $\mathcal{R}^0 = 0$ is the trivial module.

Passing to the *second equality* (20), the reasonings will be applied to "modulo Z_1" as follows. We abbreviate $Y = Z_2$ and consider the inclusions

$$\cdots \supset \Omega(X)_L \supset \Omega(X)_{L-1} = \mathrm{Ker}_Y \Omega(X)_L \supset \mathrm{Ker}_Y^2 \Omega(X)_L \supset \mathrm{Ker}_Y^3 \Omega(X)_L \supset \cdots$$

with L large enough. The strict inclusions again terminate and we obtain the new filtration

$$\begin{aligned}
\bar{\Omega}(X)_* : \bar{\Omega}(X)_0 &= \mathrm{Ker}_Y^{K-1} \Omega(X)_L \subset \bar{\Omega}(X)_1 = \mathrm{Ker}_Y^{K-2} \Omega(X)_L \subset \cdots \\
&\subset \bar{\Omega}(X)_{K-2} = \mathrm{Ker}_Y \Omega(X)_L \subset \bar{\Omega}(X)_{K-1} = \Omega(X)_L \subset \bar{\Omega}(X)_K = \Omega(X)_{L+1} \subset \cdots
\end{aligned} \tag{22}$$

of diffiety Ω with strict inclusions together with the module

$$\mathcal{R}^1 = \mathrm{Ker}_Y^K \Omega(X)_L = \mathrm{Ker}_Y \bar{\Omega}(X)_0 \subset \bar{\Omega}(X)_0 \subset \Omega.$$

Theorem 4.2 ([8],[9])**.** *Filtration* (22) *does not depend on the choice of Y. The module $\mathcal{R}^1$ is flat and does not depend even on the choice of the original good filtration Ω_*.*

We may denote $\mathcal{R}^1 = \mathcal{R}^1(\Omega)$ and there are equalities

$$\mathrm{Ker}_Y \bar{\Omega}(X)_{l+1} = \bar{\Omega}(X)_l \ (l > 0), \ \mathrm{Ker}_Y \bar{\Omega}(X)_0 = \mathcal{R}^1, \ \mathrm{Ker}_Y \mathcal{R}^1 = \mathcal{R}^1$$

corresponding to the injections

$$Y : \bar{\mathcal{M}}[1]_l \to \bar{\mathcal{M}}[1]_{l+1} \ (l > 0), \ \bar{\mathcal{M}}[1]_0 / \mathcal{R}^1 \to \bar{\mathcal{M}}[1]_1 \ (\bar{\mathcal{M}}[1]_l = \bar{\Omega}(X)_l / \bar{\Omega}(X)_{l-1})$$

for all not too special vector fields $Y \in \mathcal{H}$. The second equality (20) is universal if and only if $\mathcal{R}^1 = 0$ is the trivial module.

The procedure can be continued with $Z_3, Z_4, \ldots$ as well with quite analogous result. We obtain certain filtrations

$$\bar{\Omega}_*, \bar{\Omega}(Z_1)_*, \bar{\Omega}(Z_1, Z_2)_*, \ldots$$

which are *good* in the common sense that

$$\mathcal{L}_\mathcal{H} \bar{\Omega}(\cdot)_l \subset \bar{\Omega}(\cdot)_l \ (all \ l), \quad \bar{\Omega}(\cdot)_l + \mathcal{L}_\mathcal{H} \bar{\Omega}(\cdot)_l = \bar{\Omega}(\cdot)_{l+1} \ (l \text{ large enough}) \tag{23}$$

and moreover *standard* in the sense that

$$\begin{aligned}
Ker_{Z_{k+1}} \bar{\Omega}(Z_1, \ldots, Z_k)_{l+1} &= \bar{\Omega}(Z_1, \ldots, Z_k)_l \quad (l > 0), \\
Ker_{Z_{k+1}} \bar{\Omega}(Z_1, \ldots, Z_k)_0 &= \mathcal{R}^k, \ Ker_{Z_{k+1}} \mathcal{R}^k = \mathcal{R}^k.
\end{aligned} \tag{24}$$

The procedure becomes trivial if $k > \nu = \nu(\Omega)$ since then

$$\Omega(Z_1, \ldots, Z_k)_l = \Omega \ (l \text{ large enough}), \ \bar{\Omega}(Z_1, \ldots, Z_k)_l = \Omega, \ \mathcal{R}^k = \Omega.$$

The resulting *residual submodules* $\mathcal{R}^k \subset \Omega$ constitute the *controllability series*

$$\mathcal{R}^0 \subset \mathcal{R}^1 \subset \cdots \subset \mathcal{R}^{\nu+1} = \Omega, \tag{25}$$

to be discussed below in more detail. We speak of a *controllable diffiety* Ω if $\mathcal{R}^0 = \cdots = \mathcal{R}^\nu = 0$ are trivial modules.

5 Ordinary Differential Equations

We interrupt the general theory in order to clearly demonstrate the substance of the above constructions. Let us mention the relatively simple diffieties $\Omega \subset \Phi(\mathbf{M})$ with one independent variable $x = x_1$ (abbreviation). Let $\Omega_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega$ be a good filtration. We recall the mappings

$$\mathcal{L}_D : \Omega_l \to \Omega_{l+1}, \ Z : \mathcal{M}_l \to \mathcal{M}_{l+1} \qquad (\mathcal{M}_l = \Omega_l / \Omega_{l-1}, \ Z \in \mathcal{H}).$$

Since $\mathcal{H} = \mathcal{H}(\Omega) \subset \mathcal{T}(\mathbf{M})$ is one–dimensional submodule, then, omitting the trivial subcase $\dim \mathbf{M} < \infty$, we have the Hilbert polynomial

$$\dim \mathcal{M}_l = c_0 = \mu(\Omega) > 0 \qquad (l \text{ large enough}). \tag{26}$$

The involutivity becomes trivial, however, the standard filtration (21) with any nonvanishing vector field $X = Z_1 \in \mathcal{H}(\Omega)$ is worth mentioning.

We may choose $X = D = D_1$ the total derivative defined by the properties

$$Dx = 1, \ \omega(D) = 0 \quad (\omega \in \Omega)$$

(In particular examples to follow, explicit formulae will be stated.) In accordance with general theory, let us suppose

$$\mathrm{Ker}_D \Omega_{l+1} = \Omega_l \quad (l \geq L), \qquad \mathrm{Ker}_D^k \Omega_L = \mathrm{Ker}_D^{k+1} \Omega_L \quad (k \geq K).$$

Then the standard filtration

$$\bar{\Omega}_* : \bar{\Omega}_0 = \mathrm{Ker}_D^{K-1} \Omega_L \subset \cdots \subset \bar{\Omega}_{K-2} = \mathrm{Ker}_D \Omega_L \subset \bar{\Omega}_{K-1} = \Omega_L \subset \bar{\Omega}_K = \Omega_{L+1} \subset \cdots \tag{27}$$

together with the flat submodule

$$\mathcal{R}^0 = \mathrm{Ker}_D^K \Omega_L = \mathrm{Ker}_D \bar{\Omega}_0 \subset \Omega_0 \qquad (\mathrm{Ker}_D \mathcal{R}^0 = \mathcal{R}^0) \tag{28}$$

easily appears by a merely linear algebra. (We again refer to examples for quite explicit results. Warning: the following construction of the *standard basis* cannot be directly generalized for the case of several independent variables.) In the meantime, we also obtain a rather useful basis of diffiety Ω as a by–product. This is made as follows. Let us choose a basis

$\tau^r \ (r = 1, \ldots, R = \dim \mathcal{R}^0)$ of module $\mathcal{R}^0$ and then together with
$\pi_0^j \ (j = 1, \ldots, j_0)$ basis of the module $\bar{\Omega}_0$ and then together with
$\pi_1^j = \mathcal{L}_D \pi_0^j \ (j \text{ as above}), \pi_0^{j'} \ (j' = j_0 + 1, \ldots, j_1)$ of module $\bar{\Omega}_1$, and then with
$\pi_2^j = \mathcal{L}_D^2 \pi_0^j, \pi_1^{j'} = \mathcal{L}_D \pi_0^{j'}, \pi_0^{j''} \ (j'' = j_1' + 1, \ldots, j_2')$ of module $\bar{\Omega}_2$,
$\cdots$.

The procedure in a certain sense stops. The identity

$$\mathrm{Ker}_D \bar{\Omega}_K = \mathrm{Ker}_D \Omega_{L+1} = \Omega_L = \bar{\Omega}_{K-1}$$

implies $j_K = j_{K+1}$ and analogously $j_{K+1} = j_{K+2} = \cdots$ as well. We obtain only a finite number j_k of *initial forms*

$$\pi_0^1, \ldots, \pi_0^{j_0} \in \bar{\Omega}_0; \pi^{j_0+1}, \ldots, \pi_0^{j_1} \in \bar{\Omega}_1; \ldots; \pi_0^{j_{K-1}+1}, \ldots, \pi_0^{j_K} \in \bar{\Omega}_K \tag{29}$$

with the lower zero indices and they provide the so called *standard basis*

$$\tau^r \quad (r = 1, \ldots, R), \quad \pi_s^j = \mathcal{L}_D^s \pi_0^j \quad (j = 1, \ldots, j_K; \ s = 0, 1, \ldots) \tag{30}$$

of diffiety Ω. In fact $j_K = c_0 = \mu(\Omega)$ follows from (26) and since $\mathcal{R}^0$ is flat, there exists even a basis $\tau^r = dt^r \ (r = 1, \ldots, R = \dim \mathcal{R}^0)$.

The result can be transparently visualized (see Figure 2): the original "cross–arrows $\mathcal{L}_Z$" are "collected" in $\mathcal{R}^0$ and only the infinite sequences $\pi_r^j = \mathcal{L}_D^r \pi_0^j \ (j = 1, \ldots, \mu(\Omega); \ r = 0, 1, \ldots)$ without any crossing remain.

$$\Omega_0 \quad \Omega_1 \quad \Omega_2 \quad \ldots \qquad\qquad \mathcal{R}^0 \quad \bar{\Omega}_0 \quad \bar{\Omega}_1 \quad \bar{\Omega}_2 \quad \bar{\Omega}_3 = \Omega_2 \quad \ldots$$

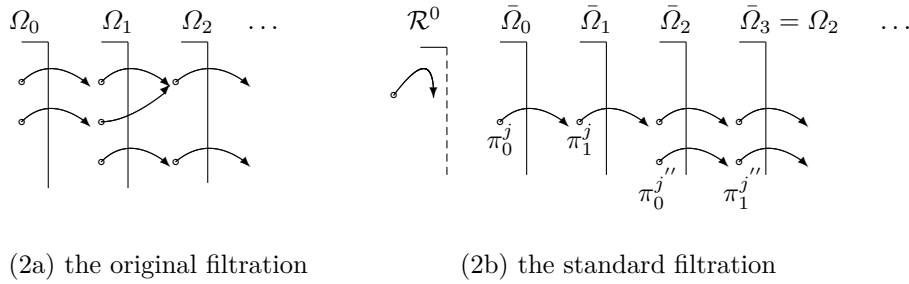(2a) the original filtration   (2b) the standard filtration

Figure 2: Filtrations.

Some immediate consequences of this construction both for the general theory and for the practice of calculations are as follows.

5.1 The Uniqueness

We recall that the module $\mathcal{R}^0$ is unique. The standard filtration (27) is unique if $\mu(\Omega) = 1$ and we refer to quite simple proof [7, Theorem 26]. This is the main result of the stimulating article [16] where the concept of the *absolute theory* was introduced for the first time. Close connection to the beautiful explicit solvability *Monge problem* [19],[20],[5] is worth mentioning, see also below.

5.2 The Morphisms [7]

If $\mathbf{m} : \mathbf{M} \to \mathbf{M}$ is a morphism of Ω then

$$D\mathbf{m}^*x \cdot \mathbf{m}^*\mathcal{L}_D\omega = \mathcal{L}_D\mathbf{m}^*\omega \quad (\omega \in \Omega)$$

and in terms of the standard basis

$$\mathbf{m}^*\tau^r \in \mathcal{R}^0, D\mathbf{m}^*x \cdot \mathbf{m}^*\pi_{s+1}^j = \mathcal{L}_D\mathbf{m}^*\pi_s^j \quad (\text{all } r, j, s). \tag{31}$$

Such a morphism is invertible ($\mathbf{m}$ is a symmetry) if and only if

$$\mathbf{m}^*\mathcal{R}^0 = \mathcal{R}^0, \pi_0^j \in \mathbf{m}^*\Omega \quad (j = 1, \ldots, \mu(\Omega)). \tag{32}$$

It is however not easy to apply these results effectively in the general equivalence theory. For instance, the solution of the symmetry problem even for the favourable case $\mu(\Omega) = 1$ in the famed article [21] was not yet undestood in full detail, see the last sentence in the prominent textbook [22].

5.3 The Variations [7]

On the contrary, the surprisingly simple explicit formula

$$Z = zD + \sum z^r \frac{\partial}{\partial t^r} + \sum D^s p^j \frac{\partial}{\partial \pi_s^j} \tag{33}$$

for all variations holds true. Here $z, p^j \in \mathcal{F}(\mathbf{M})$ are arbitrary functions and $z^r = z^r(t^1, \ldots, t^R)$ arbitrary composed functions. This is a consequence of general Lemma 2.1 applied to the standard basis (30).

5.4 The Infinitesimal Symmetries [5],[6],[7]

Variation (33) generates a Lie group if and only if it preserves an appropriate good (equivalently: standard) filtration. This is informally described in Figures 1a and 1b where the "dotted filtration" is not known in advance. In the favourable case $\mu(\Omega) = 1$, we may deal only with the unique standard filtration 1a and the dotted Figure 1b can be omitted.

5.5 A Pesimistic Prediction

Difficulties appearing in the case $\mu(\Omega) > 1$ were explicitly declared in [16]. This is the only pessimistic prognosis for a mathematical theory in E. Cartan's work and it was justified: the higher–order transformations of the absolute theory with the general Monge problem are actually forgotten.

5.6 The Jet Diffieties

Passing to more individual examples from now on, we introduce the space $\mathbf{M}(m)$ with local *jet coordinates*

$$x, \ w_s^j \quad (j = 1, \ldots, m; \ s = 0, 1, \ldots) \tag{34}$$

and the submodule $\Omega(m) \subset \Phi(\mathbf{M}(m))$ of all *contact forms*

$$\omega = \sum f_s^j \omega_s^j \quad (\text{finite sum}, f_s^j \in \mathcal{F}(\mathbf{M}(m)), \omega_s^j = \mathrm{d}w_s^j - w_{s+1}^j \mathrm{d}x) \tag{35}$$

with the total derivative

$$D = \frac{\partial}{\partial x} + \sum w_{s+1}^j \frac{\partial}{\partial w_s^j} \quad \text{where} \quad \mathcal{L}_D \omega_s^j = \omega_{s+1}^j$$

It follows that we have a diffiety: if $\Omega(m)_l \subset \Omega$ $(l = 0, 1, \ldots)$ is the submodule of the l–th *order contact forms* (with $s \leq l$ in formula (35)) then

$$\Omega(m)_* : \Omega(m)_0 \subset \Omega(m)_1 \subset \cdots \subset \Omega = \cup \Omega(m)_l \tag{36}$$

is a good filtration. This is even a standard filtration where $\mathcal{R}^0 = 0$ is trivial module and the contact forms $\omega_s^j = \pi_s^j$ provide the standard basis.

We may refer to [2] for many particular examples of automorphisms $\mathbf{m} : \mathbf{M}(m) \to \mathbf{M}(m)$ of diffiety $\Omega(m)$ involving, e.g., the above wave mechanisms. We recall that they are in general unknown.

On the contrary, all variations

$$Z = zD + \sum D^s p^j \ \frac{\partial}{\partial \omega_s^j} \quad (z, p^1, \ldots, p^m \text{ arbitrary functions}) \tag{37}$$

are well–known. They constitute a huge Lie algebra and one of the main tasks of the soliton theory concerns the determination of the special Abelian Lie subalgebras, the so called *integrable hierarchies*. On this occasion, we cannot forget the impressive monograph [23]. Variations (37) preserving the filtration (36) are either the infinitesimal point transformations or the infinitesimal Lie's contact transformations. This is a particular case of the Lie–Bäcklund theorem [10],[24]. Variations preserving other good filtrations as in Figure(1b) and therefore generating a group of higher–order transformations are in general unknown.

5.7 The Hilbert–Cartan Equation

In order to demonstrate quite explicit results, we mention the infinitesimal symmetries of the ordinary differential equation $d^2u/\mathrm{d}x^2 = F(\mathrm{d}v/\mathrm{d}x)$ with two unknown functions $u = u(x)$ and $v = v(x)$ thoroughly treated in article [7]. The equation corresponds to the diffiety $\Omega \subset \Phi(\mathbf{M})$ generated by the forms

$$\alpha_0 = du_0 - u_1\mathrm{d}x, \alpha_1 = du_1 - F(v_1)\mathrm{d}x, \beta_r = \mathrm{d}v_r - v_{r+1}\mathrm{d}x \quad (r = 0, 1, \ldots)$$

in the space $\mathbf{M}$ with coordinates x, u_0, u_1, v_r $(r = 0, 1, \ldots)$. Then

$$D = \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u_0} + F\frac{\partial}{\partial u_1} + \sum v_{r+1}\frac{\partial}{\partial v_r}$$

where

$$\mathcal{L}_D \alpha_0 = \alpha_1, \ \mathcal{L}_D \alpha_1 = D\rfloor d\alpha_1 = F'\beta_1, \ \mathcal{L}_D \beta_r = \beta_{r+1} \quad (r = 0, 1, \ldots)$$

and the submodules $\Omega_l \subset \Omega$ $(l = 0, 1, \ldots)$ consisting of the forms

$$\omega = a_0\alpha_0 + a_1\alpha_1 + \sum b_r\beta_r \qquad (r \le l; \, l = 0, 1, \ldots)$$

with arbitrary coefficients a_0, a_1, b_r determine the primary good filtration $\Omega_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega$ of diffiety Ω. Clearly $\mu(\Omega) = 1$, the standard filtration is unique and can be described as follows. Assume $DF' \ne 0$ which is satisfied if $F'' \ne 0$. Abbreviating

$$\alpha = \alpha_1 - F'\beta_0, \; \pi_0^1 = F'\alpha + DF'\alpha_0,$$

then

$$\mathcal{L}_D\alpha = -DF'\beta_0, \; \pi_1^1 = \mathcal{L}_D\pi_0^1 = 2DF'\alpha + D^2F'\alpha_0,$$

hence $\mathcal{R}^0 = 0$ is the trivial module and the submodules

$$\begin{aligned}
\bar{\Omega}_0 &= \{\pi_0^1\} = \{F'\alpha_1 + DF'\alpha_0\} \\
\bar{\Omega}_1 &= \{\pi_0^1, \pi_1^1\} = \{F'\alpha_1 + DF'\alpha_0, 2DF'\alpha\} = \{\alpha, \alpha_0\} \\
\bar{\Omega}_2 &= \{\alpha, \alpha_0, \mathcal{L}_D\alpha, \mathcal{L}_D\alpha_0\} = \{\alpha, \alpha_0, -DF'\beta_0, \alpha_1\} = \{\alpha_0, \alpha_1, \beta_0\} = \Omega_1 \\
\bar{\Omega}_3 &= \bar{\Omega}_2 + \mathcal{L}_D\bar{\Omega}_2 = \Omega_1 + \mathcal{L}_D\Omega_1 = \Omega_2, \; \bar{\Omega}_4 = \Omega_3, \; \ldots
\end{aligned}$$

of Ω constitute the (unique) standard filtration $\bar{\Omega}_*$ with the initial form π_0^1. The variations

$$Z = zD + \sum D^r p \frac{\partial}{\partial \pi_r^1} \quad (\text{arbitrary } z, p \in \mathcal{F}(\mathbf{M}(m)))$$

preserving the standard filtration are just the infinitesimal symmetries. So we have the requirement

$$\mathcal{L}_Z\pi_0^1 = Z \rfloor d\pi_0^1 + d\pi_0^1(Z) = Z \rfloor d\pi_0^1 + dp = \lambda\pi_0^1$$

with unknown factor λ. The calculations are lengthy. First of all, certain formulae [7, (107)] not stated here uniquely express z and λ in terms of the function p. Then the crucial equation

$$p = F'^2 P(\cdot) + F'Q(\cdot)v_2 \quad ((\cdot) = (x, u_0, u_1, v_0, v_1)),$$

where

$$P = \frac{1}{F'}(Q_x + u_1 Q_{u_0}) + \int \left(\frac{1}{F'}\right)' F dv_1 \cdot Q_{u_1} + \int \left(\frac{1}{F'}\right)' v_1 dv_1 \cdot Q_{v_0} + \bar{P}(x, u_0, u_1, v_0)$$

can be derived, see [7, (133)]. In the *generic subcase* [7, (144)] we obtain the final solution

$$Q = (Ax + \bar{A})u_1 + Bv_0 + C_1 x - 2Au_0 + C_3, \bar{P} = Av_0 + C \quad (A, \ldots, C \in \mathbb{R})$$

but special functions F admit more symmetries. For instance the exceptional 14–dimensional Lie algebra $\mathbb{G}_2$ of symmetries for the *Hilbert–Cartan equation* where $F = (dv/dx)^{1/2}$ was obtained [7, (175)] by for the first time direct calculations in full accordance with quite other approach in the article [21].

5.8 Example: the Monge Equation

We mention the differential equation $dw/dx = F(du/dx, dv/dx)$ with nonconstant F, however, only some conceptual topics will be discussed and we refer to [7] for more detailed survey. The corresponding diffiety $\Omega \subset \Phi(\mathbf{M})$ has the natural basis denoted

$$\alpha_r = du_r - u_{r+1}dx, \beta_r = dv_r - v_{r+1}dx \quad (r = 0, 1, \ldots), \gamma = dw - F(u_1, v_1)dx$$

in the space $\mathbf{M}$ with coordinates x, u_r, v_r $(r = 0, 1, \ldots)$ and w. The forms α_r, β_r $(r \le l)$ and γ constitute a basis of module Ω_l $(l = 0, 1, \ldots)$ of the original good filtration Ω_*. The obvious identity

$$\mathcal{L}_D\gamma = F_{u_1}\alpha_1 + F_{v_1}\beta_1 \quad (D = \frac{\partial}{\partial x} + \sum u_{r+1}\frac{\partial}{\partial u_r} + \sum v_{r+1}\frac{\partial}{\partial v_r} + F\frac{\partial}{\partial w} \in \mathcal{H})$$

represents the "cross" in Figure 2a. Then $\mathcal{R}^0 = 0$ and the forms

$$\pi_0^1 = \gamma - F_{u_1}\alpha_0 - F_{v_1}\beta_0, \ \pi_0^2 = \beta_0$$

are initial for the standard filtration of Figure 2b. In more detail, the formulae

$$\mathcal{L}_D\alpha_r = \alpha_{r+1}, \mathcal{L}_D\beta_r = \beta_{r+1} \quad (r = 0, 1, \ldots), \mathcal{L}_D\gamma = F_{u_1}\alpha_1 + F_{v_1}\beta_1$$

imply

$$\pi_1^1 = \mathcal{L}_D\pi_0^1 = -DF_{u_1}\alpha_0 - DF_{v_1}\beta_0$$

whence the submodules

$$\begin{aligned}
\bar{\Omega}_0 &= \{\pi_0^1, \pi_0^2\} = \{\gamma - F_{u_1}\alpha_0, \beta_0\}, \\
\bar{\Omega}_1 &= \bar{\Omega}_0 + \mathcal{L}_D\bar{\Omega}_0 = \{\pi_0^1, \pi_0^2, \pi_1^1, \pi_1^2\} = \{\alpha_0, \beta_0, \gamma\} = \Omega_0, \\
\bar{\Omega}_2 &= \bar{\Omega}_1 + \mathcal{L}_D\bar{\Omega}_1 = \Omega_1, \ \bar{\Omega}_3 = \Omega_2, \ldots
\end{aligned}$$

of module Ω constitute a good filtration. The alternative basis

$$\pi_0^1 = \gamma + \cdots, \ \pi_1^1 = -DF_{u_1}\alpha_0 + \cdots, \ \pi_2^1 = -DF_{u_1}\alpha_1 + \cdots, \ \cdots, \ \pi_r^2 = \beta_r \ (r \geq 0)$$

of diffiety Ω is better adapted for the calculations than the original one thanks to the absence of "crossings".

Let us start with *morphisms* $\mathbf{m} : \mathbf{M} \to \mathbf{M}$ of diffiety Ω. They are determined by formulae

$$\mathbf{m}^*\pi_0^1 = \sum a_r^j\pi_r^j, \ \mathbf{m}^*\pi_0^2 = \sum b_r^j\pi_r^j \quad \text{(arbitrary coefficients)}$$

since the remaining forms

$$\mathbf{m}^*\pi_r^k = \mathbf{m}^*\mathcal{L}_D^r\pi_0^k \quad (k = 1, 2; \ r = 1, 2, \ldots)$$

satisfy the recurrence (31). We are however interested in *invertible morphisms* $\mathbf{m}$ though the criterion (32) provides rather strong additional condition for the coefficients a_r^j and b_r^j. The problem is not yet completely resolved.

On the contrary, all *variations*

$$Z = zD + \sum D^r p \ \frac{\partial}{\partial\pi_r^1} + \sum D^r q \ \frac{\partial}{\partial\pi_r^2} \quad \text{(arbitrary } z, p, q)$$

are given by simple explicit formulae. We are however interested also in *infinitesimal symmetries Z* and they moreover are bound to preserve a certain good filtration (Figures 1a and 1b). Since $\mu(\Omega) = 2$, there are many possibilities and this is the reason why the symmetry problem for our seemingly simple Monge equation is extremely difficult. We can only refer to three particular examples of symmetries

$$\mathcal{L}_Z\pi_0^1 = \mu\pi_0^1, \ \mathcal{L}_Z\pi_0^2 = \lambda_0^1\pi_0^1 + \lambda_0^2\pi_0^2 + \lambda_1^1\pi_1^1,$$

$$\mathcal{L}_Z\pi_0^1 = \mu_0^1\pi_0^1 + \mu_0^2\pi_0^2 + \mu_1^2\pi_1^2, \ \mathcal{L}_Z\pi_0^2 = \mu\pi_0^2,$$

$$\mathcal{L}_Z\pi_0^1 = \lambda^1\pi_0^1 + \lambda^2\pi_0^2, \ \mathcal{L}_Z\pi_0^2 = \mu^1\pi_0^1 + \mu^2\pi_0^2$$

discussed in [7] especially for the case $F = u_1v_1$.

5.9　The Monge Problem

There exists an automorphism $\mathbf{m} : \mathbf{M}(3) \to \mathbf{M}(3)$ of the jet diffiety $\Omega(3)$ such that

$$\mathbf{m}^*(x - F(w_0^1, w_0^2)) = w_1^3 - F(w_1^1, w_1^2)$$

for every nonconstant function F, see [7, Appendix] and this may be regarded as a solution of a Monge problem, which is as follows.

We recall coordinates x, w_s^j $(j = 1, 2, 3;\ s = 0, 1, \dots)$ of the space $\mathbf{M}(3)$ and the diffiety $\Omega(3)$ generated by all contact forms $\omega_s^j = \mathrm{d}w_s^j - w_{s+1}^j \mathrm{d}x$ where the solutions are given by formulae

$$w_s^j = w_s^j(x) = \frac{\mathrm{d}^s w^j(x)}{\mathrm{d}x^s} \qquad \text{(arbitrary functions } w^j(x);\ j = 1, 2, 3\text{)}.$$

(In traditional terms, we deal with all curves in four–dimensional space with variables x, w^1, w^2, w^3.) Let us consider only such curves that moreover satisfy

$$x = F(w^1(x), w^2(x)).$$

(Informally, they are lying in the three–dimensional subspace $x = F(w^1, w^2)$. In fact we have the lower diffiety $\Omega(2)$.) The mentioned automorphism $\mathbf{m}$ identifies these curves with all solutions of the Monge equation

$$\frac{\mathrm{d}w_0^3}{\mathrm{d}x} = F\left(\frac{\mathrm{d}w_0^1}{\mathrm{d}x}, \frac{\mathrm{d}w_0^2}{\mathrm{d}x}\right).$$

(Altogether, there is explicit correspondence between solutions of diffiety $\Omega(2)$ and solutions of the Monge equation.)

Still, in other terms, we have the subspace $\mathbf{N} \subset \mathbf{M}(3)$ given by equations

$$D^r(x - F(w_0^1, w_0^2)) = 0 \quad (r = 0, 1, \dots;\ D = \frac{\partial}{\partial x} + \sum w_{r+1}^j \frac{\partial}{\partial w_r^j})$$

which is clearly isomorphic to $\mathbf{M}(2)$. It follows that *the corresponding jet diffiety $\Omega(2) \subset \Phi(\mathbf{M}(2))$ is isomorphic to the diffiety Ω of the Monge equation.* Roughly saying, the diffiety Ω of the Monge equation is in fact the "trivial" diffiety $\Omega(2)$.

5.10 Calculus of Variations [5],[7],[25],[26],[27],[28]

The classical Lagrange problem concerning the one–dimensional variational integral subject to differential constraints is represented by a diffiety $\Omega \subset \Phi(\mathbf{M})$ together with a given form $\varphi \in \Phi(\mathbf{M})$. We are interested in the *variational integrals*

$$\int_a^b \mathbf{n}^* \varphi \qquad (\mathbf{n} : \mathbf{N} \to \mathbf{M},\ \mathbf{n}^* \Omega = 0,\ \mathbf{N} : a \le x \le b \subset \mathbb{R})$$

evaluated for the solutions $\mathbf{n} : \mathbf{N} \to \mathbf{M}$ of diffiety Ω. Such a solution is called *extremal* if $\mathbf{n}^* Z \rfloor d\varphi = 0$ for all variations Z along $\mathbf{N}$, that is, $\mathbf{n}^* \mathcal{L}_Z \Omega = 0$. This is in full accordance with the common approach since then the obvious identities

$$\mathbf{n}^* \mathcal{L}_Z \varphi = \mathrm{d}\, \mathbf{n}^* \varphi(Z),\ \int_a^b \mathbf{n}^* \mathcal{L}_Z \varphi = \mathbf{n}^* \varphi(Z)|_{x=a}^{x=b}$$

declare that the variation of the integral (in the common sense) indeed depends only on the boundary values. One can observe that the form φ can be replaced with any form $\varphi + \omega$ $(\omega \in \Omega)$ without change of the extremals. At this place, let us apply the standard filtrations. Assuming the controllability $\mathcal{R}^0 = 0$, a unique *Poincaré–Cartan form* $\breve{\varphi} = \varphi + \breve{\omega}$ with appropriate $\breve{\omega} \in \Omega$ exists such that

$$\mathrm{d}\breve{\varphi} \cong \sum e^j \pi_0^j \wedge \mathrm{d}x \quad (\mathrm{mod}\ \Omega \wedge \Omega;\ j = 1, \dots, \mu(\Omega)). \tag{38}$$

This implies that the extremals $\mathbf{n}$ are characterized by the *Euler–Lagrange equations* $\mathbf{n}^* e^j = 0$ $(j = 1, \dots, \mu(\Omega))$ and satisfy the identities

$$\mathbf{n}^* Z \rfloor \mathrm{d}\breve{\varphi} = 0,\ \int_a^b \mathbf{n}^* \mathcal{L}_Z \breve{\varphi} = \mathbf{n}^* \breve{\varphi}(Z)|_{x=a}^{x=b}$$

for *all vector fields $Z \in \mathcal{T}(\mathbf{M})$*.

In particular $e^j = 0$ $(j = 1, 2, \dots)$ identically if and only if the Poincaré–Cartan form $\breve{\varphi}$ is a total differential of appropriate function g hence $\varphi \cong \breve{\varphi} \cong Dg \cdot \mathrm{d}x \pmod{\Omega}$. The *Noether theorem* immediately follows as well. Assuming

$$\mathcal{L}_Z \Omega \subset \Omega, \ \mathcal{L}_Z \varphi \in \Omega \qquad \text{(appropriate } Z \in \mathcal{T}(\mathbf{M})), \tag{39}$$

the function $\mathbf{n}^* Z \rfloor \mathrm{d}\breve{\varphi}$ is clearly constant for every extremal $\mathbf{n}$.

5.11   Example: a Variational Integral

We mention only the variational integral

$$\int f(x, u_0, v_0, w_0, u_1, v_1) \mathrm{d}x \qquad (u_r = \frac{\mathrm{d}^r u}{\mathrm{d}x^r}, v_r = \frac{\mathrm{d}^r v}{\mathrm{d}x^r})$$

with the constraint

$$w_1 = F(x, u_0, v_0, w_0, u_1, v_1).$$

Assuming

$$a = F_{v_0} - DF_{v_1} + F_{w_0} F_{v_1} \neq 0 \quad (D = \frac{\partial}{\partial x} + \sum u_{r+1} \frac{\partial}{\partial u_r} + \sum v_{r+1} \frac{\partial}{\partial v_r} + F \frac{\partial}{\partial w_0}),$$

there exists the Poincaré–Cartan form

$$\breve{\varphi} = f \mathrm{d}x + (f_{u_1} - \frac{b}{a} F_{u_1})\alpha + (f_{v_1} - \frac{b}{a} F_{v_1})\beta - \frac{b}{a}\gamma$$

where

$$b = f_{v_0} - Df_{v_1} + f_{w_0} F_{v_1}, \alpha = du_0 - u_1 \mathrm{d}x, \beta = \mathrm{d}v_0 - v_1 \mathrm{d}x, \gamma = \mathrm{d}w_0 - F\mathrm{d}x$$

and the Euler–Lagrange equations

$$e^1 = f_{w_0} - \frac{b}{a} F_{w_0} - D\frac{b}{a} = 0, \ e^2 = B - \frac{b}{a} A = 0$$

where

$$A = F_{u_0} - DF_{u_1} + F_{w_0} F_{u_1}, \ B = f_{u_0} - Df_{u_1} + f_{w_0} F_{u_1}.$$

Since no uncertain multipliers appear, the Legendre, Jacobi, Hilbert–Weierstrass extremality conditions, the Hamilton–Jacobi equations and the geodesic fields can be investigated without much difficulty quite analogously as in the traditional unconstrained theory [25],[26],[27],[28].

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