Two Parameter Families of Global Solutions for Some Classes of Nonlinear Second Order Ordinary Differential Equations

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Abstract The Riccati equation method is used to establish some criteria for the existence of two-parameter families of global solutions for some classes of second order non-linear ordinary differential equations. Two oscillation theorems are proved. The results are applied to the Emden - Fowler equation and to the Van der Pol type equation.

Keywords: Riccati equation, global solvability, oscillation, singular oscillation, Emden - Fowler equation, conditional stability, Van der Pol type equation.

1 Introduction

Due to numerous applications the nonlinear ordinary differential equations occupy an important place in the theory of differential equations and numerous works are devoted to them (see [1] and cited works in it, [2 - 8]). Except in rare cases these equations cannot be integrated explicitly. Therefore the study of such characteristics of their solutions as global existence, oscillation, asymptotic behavior, stability and plays an important role.

Let $p_0(t; w)$, $q_0(t; w)$ and $r_0(t; w)$ be continuous on $[t_0; +\infty) \times (-\infty; +\infty)$ real valued functions, and let $p_0(t; w) > 0$, $t \geq t_0$, $w \in (-\infty; +\infty)$. Consider the equation

$$(p_0(t; \phi(t))\phi'(t))' + q_0(t; \phi(t))\phi'(t) + r_0(t; \phi(t))\phi(t) = 0, \qquad t \geq t_0. \qquad (1.1)$$

Like the linear differential equations of the second order, this equation can be interpreted as a system of nonlinear differential equations of the first order (see [3], p. 381):

$$\begin{cases} \phi'(t) = \frac{\psi(t)}{p_0(t; \phi(t))}; \\[2mm] \psi'(t) = -r_0(t; \phi(t))\phi(t) - \frac{q_0(t; \phi(t))}{p_0(t; \phi(t))}\psi(t). \end{cases} \qquad (1.2)$$

By Peano's theorem (see [3], p. 21, 22) for every $\phi_{(0)}$ and $\phi_{(1)}$ (here and henceforth $\phi_{(0)}$ and $\phi_{(1)}$ are real numbers) and $t_1 \geq t_0$ the system (1.2) has a solution $(\phi(t), \psi(t))$ in the neighborhood of the point t_1 (in the case $t_1 = t_0$ in some right neighborhood of t_0), satisfying the initial conditions: $\phi(t_1) = \phi_{(0)}$, $\psi(t_1) = \phi_{(1)}$. Therefore, for any $\phi_{(0)}$, $\phi_{(1)}$ and $t_1 \geq t_0$ eq. (1.1) has a solution in some neighborhood of the point t_1, satisfying the initial conditions: $\phi(t_1) = \phi_{(0)}$, $\phi'(t_1) = \phi_{(1)}$.

Remark 1.1. The solution $\phi(t)$ of eq. (1.1), satisfying the initial conditions: $\phi(t_1) = \phi_{(0)}$, $\phi'(t_1) = \phi_{(1)}$ in general, is not the unique. However, under additional restrictions on the functions $p_0(t; w), q_0(t; w)$ and $r_0(t; w)$ it is the unique. For example, due to (1.2), if the functions $f_1(t; u; v) \equiv \frac{v}{p_0(t;u)}$, $f_2(t; u; v) \equiv r_0(t; u)u + \frac{q_0(t;u)}{p_0(t;u)}v$ satisfy the Lipschitz condition jointly u, v in the region $O \equiv \{(t; u; v) : |t - t_1| \leq \delta, |u - \phi_{(0)}| \leq M, |v - \phi_{(1)}| \leq N\}$, $\delta > 0$, $M > 0$, $N > 0$ (see [3]. p. 13), then by virtue of the Picard - Lindellef's theorem (see [3], p. 19) the solution $\phi(t)$ of eq. (1.1) with $\phi(t_1) = \phi_{(0)}$, $\phi'(t_1) = \phi_{(1)}$ exists on the interval $[t_1; t_2]$ and is the unique, where $t_2 \equiv \min\{\delta, \frac{\sqrt{M^2+N^2}}{M_0}\}, M_0 \equiv \max_{(t;u;v) \in O} \sqrt{f_1^2(t; u; v) + f_2^2(t; u; v)}$ (as far as the solution $(\phi(t), \psi(t))$ of the system (1.2) with $\phi(t_1) = \phi_{(0)}$, $\psi(t_1) = \phi_{(1)}$ exists on the interval $[t_1; t_2]$).

Example 1.1. For $p_0(t;u) \equiv |u|^\sigma, \sigma > 0, \quad q_0(t;u) \equiv 0, \quad r_0(t;u) \equiv |u|^\nu, \quad \nu < 0$ the functions $f_j(t;u;v), \quad j = 1,2$, satisfy the Lipschtz condition in the region $|t - t_1| \le \delta, \quad \delta > 0, \quad |u - 1| \le \frac{1}{2}, \quad |v| \le 1$, but not in the region $|t - t_1| \le \delta, \quad \delta > 0, \quad |u - 1| \le 2, \quad |v| \le 1$.

Example 1.2. For $p_0(t;u) \equiv 1 + t^2 + u^4, \quad q_0(t;u) \equiv t + u^3, \quad r_0(t;u) \equiv t^3 - u$ the functions $f_j(t;u;v), \quad j = 1,2$, satisfy the Lipschtz condition in the region $|t - t_1| \le \delta, \quad |u| \le M, \quad |v| \le N, \quad \delta > 0, \quad M > 0, \quad N > 0$.

In this paper the Riccati equation method is applied to establish some criteria of existence of two-parameter family of global solutions of the equation (1.1). Two oscillatory theorems are proved. The obtained results are applied to Emden - Fowler's equation and to the Van der Pol type equation.

2   Auxiliary Propositions

Throughout this paragraph let $\phi_0(t)$ be a solution of eq.(1.1) on the interval $[t_0; T) \quad (T \le +\infty); \quad y_0(t) \equiv p_0(t; \phi_0(t)) \frac{\phi_0'(t)}{\phi_0(t)}$. Consider the Riccati equation

$$y'(t) + \frac{y^2(t)}{p_0(t; \phi_0(t))} + \frac{q_0(t; \phi_0(t))}{p_0(t; \phi_0(t))} y(t) + r_0(t; \phi_0(t)) = 0, \qquad t \in [t_0; T). \tag{2.1}$$

Let $\phi_0(t) \ne 0, \quad t \in [t_1; t_2](\subset [t_0; T))$. It is not difficult to check, that the function $y_0(t)$ is a solution of eq. (2.1) on the interval $[t_1; t_2]$. We have:

$$\phi_0(t) = \phi_0(t_1) \exp\left\{ \int_{t_1}^t \frac{\phi_0'(\tau)}{\phi_0(\tau)} d\tau \right\} = \phi_0(t_1) \exp\left\{ \int_{t_1}^t \frac{y_0(\tau)}{p_0(\tau; \phi_0(\tau))} d\tau \right\}, \qquad t \in [t_1; t_2]. \tag{2.2}$$

Consider the linear equation

$$z'(t) + \frac{y_0(t) + q_0(t; \phi_0(t))}{p_0(t; \phi_0(t))} z(t) + r_0(t; \phi_0(t)) = 0, \qquad t \in [t_1; t_2]. \tag{2.3}$$

By virtue of the Cauchy's formula the general solution of this equation on the interval $[t_1; t_2)$ is given by the formula

$$z(t) = c \exp\left\{ -\int_{t_1}^t \frac{y_0(\tau) + q_0(\tau; \phi_0(\tau))}{p_0(\tau; \phi_0(\tau))} d\tau \right\} -$$

$$-\int_{t_1}^t \exp\left\{ -\int_\tau^t \frac{y_0(s) + q_0(s; \phi_0(s))}{p_0(s; \phi_0(s))} ds \right\} r_0(\tau; \phi_0(\tau)) d\tau, \quad c = const. \tag{2.4}$$

By (2.1) $y_0(t)$ is a solution of (2.3). Therefore from (2.4) it follows:

$$y_0(t) = y_0(t_1) \exp\left\{ -\int_{t_1}^t \frac{y_0(\tau) + q_0(\tau; \phi_0(\tau))}{p_0(\tau; \phi_0(\tau))} d\tau \right\} -$$

$$-\int_{t_1}^t \exp\left\{ -\int_\tau^t \frac{y_0(s) + q_0(s; \phi_0(s))}{p_0(s; \phi_0(s))} ds \right\} r_0(\tau; \phi_0(\tau)) d\tau, \qquad t \in [t_1; t_2]. \tag{2.5}$$

By (1.1) the following equality holds

$$[p_0(t; \phi_0(t)) \phi_0'(t)]' + \frac{q_0(t; \phi_0(t))}{p_0(t; \phi_0(t))} [p_0(t; \phi_0(t)) \phi_0'(t)] + r_0(t; \phi_0(t)) \phi_0(t) = 0, \qquad t \ge t_0.$$

Therefore,

$$p_0(t; \phi_0(t))\phi_0'(t) = p_0(t_1; \phi_0(t_1))\phi_0'(t_1) \exp\left\{ - \int_{t_1}^{t} \frac{q_0(\tau; \phi_0(\tau))}{p_0(\tau; \phi_0(\tau))} d\tau \right\} -$$

$$- \int_{t_1}^{t} \exp\left\{ - \int_{\tau}^{t} \frac{q_0(s; \phi_0(s))}{p_0(s; \phi_0(s))} ds \right\} r_0(\tau; \phi_0(\tau))\phi_0(\tau) d\tau, \qquad t \in [t_0; T). \tag{2.6}$$

Dividing both sides of this equality on $p_0(t; \phi_0(t))$ and integrating from t_1 to t we obtain:

$$\phi_0(t) = \phi_0(t_1) + p_0(t_1; \phi_0(t_1))\phi_0'(t_1) \int_{t_1}^{t} \exp\left\{ - \int_{t_1}^{\tau} \frac{q_0(s; \phi_0(s))}{p_0(s; \phi_0(s))} ds \right\} \frac{d\tau}{p_0(\tau; \phi_0(\tau))} -$$

$$- \int_{t_1}^{t} \frac{d\tau}{p_0(\tau; \phi_0(\tau))} \int_{t_1}^{\tau} \exp\left\{ - \int_{s}^{\tau} \frac{q_0(\xi; \phi_0(\xi))}{p_0(\xi; \phi_0(\xi))} d\xi \right\} r_0(s; \phi_0(s))\phi_0(s) ds. \tag{2.7}$$

Let $p_1(t; w)$, $q_1(t; w)$ and $r_1(t; w)$ be real valued continuous functions on the $[t_0; +\infty) \times (-\infty; +\infty)$, and let $p_1(t; w) > 0$, $t \geq 0$, $w \in (-\infty; +\infty)$. Along with the (1.1) consider the equation

$$(p_1(t; \phi(t))\phi'(t))' + q_1(t; \phi(t))\phi'(t) + r_1(t; \phi(t))\phi(t) = 0, \qquad t \geq t_0. \tag{2.8}$$

Throughout this paragraph let $\phi_1(t)$ be a solution of eq.(2.8) on the interval $[t_0; T)$ $(T \leq \leq +\infty)$; $y_1(t) \equiv p_1(t; \phi_1(t))\frac{\phi_1'(t)}{\phi_1(t)}$. Consider the Riccati equation

$$y'(t) + \frac{y^2(t)}{p_1(t; \phi_1(t))} + \frac{q_1(t; \phi_1(t))}{p_1(t; \phi_1(t))} y(t) + r_1(t; \phi_1(t)) = 0, \qquad t \in [t_0; T). \tag{2.9}$$

Let $\phi_1(t) \neq 0$, $t \in [t_1; t_2)$ $(\subset [t_0; T)$. Then as in the case of eq. (2.1) the function $y_1(t)$ is a solution of eq. (2.9) on the interval $[t_1; t_2)$ and

$$\phi_1(t) = \phi_1(t_1) \exp\left\{ \int_{t_1}^{t} \frac{y_1(\tau)}{p_1(\tau; \phi_1(\tau))} d\tau \right\}, \qquad t \in [t_1; t_2). \tag{2.10}$$

Since $y_0(t)$ and $y_1(t)$ are solutions of eq. (2.1) and eq. (2.9) respectively, we have:

$$y_j'(t) + \frac{y_j^2(t)}{p_j(t; \phi_j(t))} + \frac{q_j(t; \phi_j(t))}{p_j(t; \phi_1(t))} y_j(t) + r_j(t; \phi_j(t)) = 0, \qquad j = 0, 1.$$

Therefore,

$$[y_1(t) - y_0(t)]' + \frac{y_0(t) + y_1(t) + q_{1-j}(t; \phi_{1-j}(t))}{p_{1-j}(t; \phi_{1-j}(t))}[y_1(t) - y_0(t)] +$$

$$+ \left[\frac{1}{p_1(t; \phi_1(t))} - \frac{1}{p_0(t; \phi_0(t))}\right] y_j^2(t) + \left[\frac{q_1(t; \phi_1(t))}{p_1(t; \phi_1(t))} - \frac{q_0(t; \phi_0(t))}{p_0(t; \phi_0(t))}\right] y_j(t) +$$

$$+ r_1(t; \phi_1(t)) - r_0(t; \phi_0(t)) = 0, \qquad t \in [t_1; t_2), \ j = 0, 1. \tag{2.11}$$

By (2.5) from (2.11) we have:

$$y_1(t) - y_0(t) = [y_1(t_1) - y_0(t_1)] \exp\left\{ - \int_{t_1}^{t} \frac{y_0(\tau) + y_1(\tau) + q_{1-j}(\tau; \phi_{1-j}(\tau))}{p_{1-j}(\tau; \phi_{1-j}(\tau))} d\tau \right\} -$$

$$-\int\limits_{t_1}^{t} \exp\left\{-\int\limits_{\tau}^{t} \frac{y_0(s)+y_1(s)+q_{1-j}(s;\phi_{1-j}(s))}{p_{1-j}(s;\phi_1(s))}ds\right\}\left[\left(\frac{1}{p_1(\tau;\phi_1(\tau))}-\frac{1}{p_0(\tau;\phi_0(\tau))}\right)y_j^2(\tau)+\right.$$

$$\left.+\left(\frac{q_1(\tau;\phi_1(\tau))}{p_1(\tau;\phi_1(\tau))}-\frac{q_0(\tau;\phi_0(\tau))}{p_0(\tau;\phi_0(\tau))}\right)y_j(\tau)+r_1(\tau;\phi_1(\tau))-r_0(\tau;\phi_0(\tau))\right]d\tau, \quad j=0,1. \qquad (2.12_j)$$

Lemma 2.1. Let $y_0(t_1) \geq 0$, and let the inequality

$$r_0(t;w) \leq 0 \quad \text{for} \quad 0 < w \leq Y_0(t), \quad t \in [t_1;t_2), \qquad (2.13)$$

holds, where $Y_0(t) \equiv \max\limits_{\xi \in [t_1;t]} |\phi_0(\xi)|$. Then

$$y_0(t) \geq 0, \qquad t \in [t_1;t_2), \qquad (2.14)$$

and if $y_0(t_1) > 0$, then

$$y_0(t) > 0, \qquad t \in [t_1;t_2), \qquad (2.15)$$

Proof. It follows from (2.13), that $r_0(t,\phi(t)) \leq 0, \ t \in [t_1;t_2)$. Then

$$\int\limits_{t_1}^{t} \exp\left\{-\int\limits_{\tau}^{t} \frac{y_0(s)+q_0(s,\phi_0(s))}{p_0(s,\phi_0(s))}ds\right\}r_0(\tau,\phi_0(\tau))d\tau \leq 0, \ \ t \in [t_1;t_2).$$

From here, from (2.5) and from the inequality $y_0(t_1) \geq 0$ ($y_0(t_1) > 0$) it follows (2.14) ((2.15)). The proof of the lemma is complete.

Definition 2.1. We say, that $[t_1;t_2)$ is maximum interval of existence for the solution $\phi_0(t)$ ($y_0(t)$, $\phi_1(t)$, $y_1(t)$), if $\phi_0(t)$ ($y_0(t)$, $\phi_1(t)$, $y_1(t)$) exists on the interval $[t_1;t_2)$ and cannot be continued to right of t_2 as a solution of eq. (1.1) ((2.1), (2.8), (2.9)).

Lemma 2.2. Let $y_0(t) \geq 0, \ t \in [t_1;t_2)$, and let

$$\int\limits_{t_1}^{t_2} \frac{y_0(\tau)}{p_0(\tau;\phi_0(\tau))}d\tau < +\infty. \qquad (2.16)$$

Then $[t_1;t_2)$ is not maximum interval of existence for $y_0(t)$.

Proof. Since $y_0(t)$ is nonnegative, it follows from (2.2) and (2.16), that there exists finite limit

$$\lim_{t\to t_2-0}\phi_0(t) = \phi_0(t_1)\exp\left\{\lim_{t\to t_2-0}\int\limits_{t_1}^{t}\frac{y_0(\tau)}{p_0(\tau;\phi_0(\tau))}d\tau\right\} \neq 0. \qquad (2.17)$$

By (2.6) from here and from continuity of $p_0(t;w), \ q_0(t;w), \ r_0(t;w)$ it follows existence of finite limit $\lim\limits_{t\to t_2-0}\phi_0'(t)$. Therefore, $[t_1;t_2)$ is not maximum interval of existence for $\phi_0(t)$, and, so, by virtue of (2.17) the function $p_0(t;\phi_0(t))\frac{\phi_0'(t)}{\phi_0(t)}$ is defined on the interval $[t_1;t_2+\varepsilon)$ for some $\varepsilon > 0$. It follows from here, that $[t_1;t_2)$ is not maximum interval of existence for $y_0(t)$. The proof of the lemma is completed.

Let $u = u(t) \neq 0, \ v = v(t), \ x = x(t)$ be continuous functions on the interval $[t_0;+\infty)$,

$$I_{u,v}^{+}(t_1;t) \equiv \int\limits_{t_1}^{t}\exp\left\{-\int\limits_{t_1}^{\tau}v(s)ds\right\}\frac{d\tau}{u(\tau)}, \qquad I_{v,x}^{-}(t_1;t) \equiv \int\limits_{t_1}^{t}\exp\left\{-\int\limits_{\tau}^{t}v(s)ds\right\}x(\tau)d\tau,$$

$P(t), Q(t)$ and $R(t)$ be real valued continuous functions on the interval $[t_0;+\infty)$, and let $P(t) > 0, \ t \geq t_0$. Denote:

$$F(t_1;t;c_1;c_2) \equiv |c_1|\exp\left\{c_2 I_{P,Q}^{+}(t_1;t)-\int\limits_{t_1}^{t}I_{Q,R}^{-}(t_1;\tau)\frac{d\tau}{P(\tau)}\right\}, \ \ t_1, \ t \geq t_0.$$

Lemma 2.3. Let $y_0(t) \geq 0, \ t \in [t_1; t_2)$, and let for some $\varepsilon > 0$ the following inequalities hold:

$$p_0(t; w) \geq P(t), \quad \frac{q_0(t; w)}{p_0(t; w)} \geq Q(t), \ \ R(t) \leq r_0(t; w) \leq 0 \tag{2.18}$$

for $|w| \leq F(t_1; t; c_1; c_2) + \varepsilon, \ t \in [t_1; t_2)$, where $c_1 \equiv \phi_0(t_1) \neq 0, \ c_2 \equiv y_0(t_1)$. Then

$$\int\limits_{t_1}^{t} \frac{y_0(\tau)}{p_0(\tau; \phi_0(\tau))} d\tau \leq y_0(t_1) I^+_{P,Q}(t_1; t) - \int\limits_{t_1}^{t} I^-_{Q,R}(t_1; \tau) \frac{d\tau}{P(\tau)}, \qquad t \in [t_1; t_2). \tag{2.19}$$

Proof. Suppose, that for some $t_3 \in (t_1; t_2)$ the inequality (2.19) is not true. Then because this inequality holds for $t = t_1$, then taking into account (2.2) we get, that there exists $t_4 \in (t_1; t_3)$ such, that

$$|\phi_0(t_4)| > F(t_1; t_4; c_1; c_2); \tag{2.20};$$

$$|\phi_0(t)| \leq F(t_1; t; c_1; c_2) + \varepsilon, \qquad t \in [t_1; t_4).$$

From the last inequality and from(2.20) it follows, that

$$p_0(t; \phi_0(t)) \geq P(t), \quad \frac{q_0(t; \phi_0(t))}{p_0(t; \phi_0(t))} \geq Q(t); \ \ R(t) \leq r_0(t; \phi_0(t)) \leq 0, \qquad t \in [t_1; t_4].$$

By virtue of nonnegativity of $y_0(t)$ from here and (2.5) it follows:

$$\frac{y_0(t)}{p_0(t; \phi_0(t))} \leq \frac{c_2}{P(t)} \exp\left\{-\int\limits_{t_1}^{t} Q(\tau)d\tau\right\} - \frac{1}{P(t)} I^-_{Q,R}(t_1; t); \qquad t \in [t_1; t_4].$$

Integrate this inequality from t_1 to t_4. We obtain:

$$\int\limits_{t_1}^{t_4} \frac{y_0(\tau)}{p_0(\tau; \phi_0(\tau))} d\tau \leq c_2 I^+_{P,Q}(t_1; t_4) - \int\limits_{t_1}^{t_4} I^-_{Q,R}(t_1; \tau) \frac{d\tau}{P(\tau)}.$$

Consequently, $|\phi_0(t_4)| \leq F(t_1; t_4; c_1; c_2)$, which contradicts (2.20). The obtained contradiction proves (2.19). The proof of the lemma is complete.

Denote: $G_x(t_1; t; c_1; c_2) \equiv |c_1| \exp\left\{c_2 I^+_{P,Q}(t_1; t) + \int\limits_{t_1}^{t} \frac{x(\tau)}{P(\tau)} d\tau\right\}$.

Lemma 2.4. Let $y_0(t) \geq 0, \ t \in [t_1; t_2)$, and let for some $\varepsilon > 0$ the following inequalities hold:

$$p_0(t; w) \geq P(t), \ \ q_0(t; w) \geq Q(t) \geq 0, \ \ r_0(t; w) \leq 0, \ \ \left|p_0(t; w) \frac{r_0(t; w)}{q_0(t; w)}\right| \leq \widetilde{Q}(t) \quad \text{for} \quad |w| \leq$$

$\leq G_M(t_1; t; c_1; c_2) + \varepsilon, \ t \in [t_1; t_2)$, where $\widetilde{Q}(t)$ is a continuous function on the interval $[t_1; t_2)$, $M(t) \equiv \max\limits_{\xi \in [t_1; t]}\{\widetilde{Q}(\xi)\}, \ c_1 = \phi_0(t_1), \ c_2 = y_0(t_1)$. Then

$$\int\limits_{t_1}^{t} \frac{y_0(\tau)}{p_0(\tau; \phi_0(\tau))} d\tau \leq y_0(t_1) I^+_{P,Q}(t_1; t) - \int\limits_{t_1}^{t} \frac{M(\tau)}{P(\tau)} d\tau, \qquad t \in [t_1; t_2).$$

This lemma can be proved by analogy of the lemma 2.3. In its proof can be used the following easily verifiable inequality

$$\int\limits_{t_1}^{t} \exp\left\{-\int\limits_{\tau}^{t} \frac{y_0(s) + q_0(s; \phi_0(s))}{p_0(s; \phi_0(s))}\right\} |r_0(\tau; \phi_0(\tau))| d\tau \leq M(t), \qquad t \in [t_1; t_2).$$

Lemma 2.5. Let the following conditions hold:

a_1) $y_1(t_1) > y_0(t_1)$;

b_1) $\phi_1(t_1) \geq \phi_0(t_1) > 0$, $\phi_1'(t_1) > \phi_0'(t_1) \geq 0$ or
   $\phi_1(t_1) \leq \phi_0(t_1) < 0$, $\phi_1'(t_1) < \phi_0'(t_1) \leq 0$;

c_1) $p_0(t;w) \leq p_1(t;w_1)$, $r_0(t;w) \geq r_1(t;w_1)$, for $t \in [t_1;t_2]$, $|w| \leq |w_1|$, w,
   $w_1 \in (-\infty; +\infty)$;

d_1) $p_0(t;w) \geq p_1(t;w_1)$ for $t \in [t_1;t_2], |w| \geq |w_1|, w, w_1 \in (-\infty;+\infty)$;

e_1) $\frac{q_0(t;w)}{p_0(t;w)} \leq \frac{q_1(t;w_1)}{p_1(t;w_1)}$; $y_1(t) \geq 0$ or $y_0(t) \geq 0$ for $t \in [t_1;t_2], |w| \leq |w_1|$,
   $w, w_1 \in (-\infty; +\infty)$.

Then

$$y_1(t) > y_0(t), \qquad t \in [t_1;t_2]. \tag{2.21}$$

Proof. Suppose (2.23) is false. Then from a_1) it follows:

$$y_1(t) > y_0(t), \qquad t \in [t_1;t_3); \tag{2.22}$$

$$y_1(t_3) = y_0(t_3) \tag{2.23}$$

for some $t_3 \in (t_1;t_2)$. From b_1) it follows:

$$|\phi_1(t)| \geq |\phi_0(t)|, \qquad t \in [t_1;t_4), \tag{2.24}$$

for some $t_4 \in [t_1;t_3]$. Let us show, that

$$|\phi_0(t)| \leq |\phi_1(t)|, \qquad t \in [t_1;t_3]. \tag{2.25}$$

Suppose, that it is not so. Then it follows from (2.24), that

$$|\phi_0(t)| \leq |\phi_1(t)|, \qquad t \in [t_1;t_5], \tag{2.26}$$

$$|\phi_0(t)| > |\phi_1(t)|, \qquad t \in (t_5;t_3), \tag{2.27}$$

where $t_5 = \sup\{t \in [t_1;t_3] : |\phi_0(t)| \leq |\phi_1(t)|\} \in (t_1;t_3)$. On the strength of (2.2) and (2.10) from d_1), (2.23), (2.26) and (2.27) it follows:

$$|\phi_0(t)| = |\phi_0(t_5)| \exp\left\{\int\limits_{t_1}^{t} \frac{y_0(\tau)}{p_0(\tau;\phi_0(\tau))}\mathrm{d}\tau\right\} \leq |\phi_1(t_5)| \exp\left\{\int\limits_{t_1}^{t} \frac{y_1(\tau)}{p_1(\tau;\phi_1(\tau))}\mathrm{d}\tau\right\} = |\phi_1(t)|,$$

$t \in [t_5;t_3)$, which contradicts (2.27). The obtained contradiction proves (2.25). By (2.12) from c_1), e_1) and (2.25) it follows, that $y_1(t_3) > y_0(t_3)$, which contradicts (2.23). The obtained contradiction proves (2.21). The proof of the lemma is complete.

Remark 2.1. It follows from the conditions c_1) and d_1), that $p_0(t;w) = p_1(t;w)$, $t \in\in [t_1;t_2]$, $w \in (-\infty;+\infty)$ and $p_0(t;w)$ increases (in the wide sense) by w on the interval $[0;+\infty)$ and decreases (in the wide sense) by w on the interval $(-\infty;0]$ for every $t \in [t_1;t_2]$, in particular when $p_0(t;w) = p_1(t;w) = p(t)$, $t \in [t_1;t_2]$, then the conditions B_1) and r_1) hold.

Let us consider the Riccati equation

$$y'(t) + \frac{y^2(t)}{P(t)} + \frac{Q(t)}{P(t)}y(t) + R(t) = 0, \qquad t \geq t_0. \tag{2.28}$$

Lemma 2.6. Let the following conditions hold:

a_2) $y_0(t) \geq 0$, $t \in [t_1;t_2](\subset [t_0;T))$;

b_2) $P(t) \geq p_0(t;\phi_0(t))$, $\frac{Q(t)}{P(t)} \leq \frac{q_0(t;\phi_0(t))}{p_0(t;\phi_0(t))}$, $R(t) \leq r_0(t;\phi_0(t))$, $t \in [t_1;t_2]$.

Then the solution $y_2(t)$ of eq. (2.28), satisfying the condition $y_2(t_1) \geq y_0(t_1)$, exists on the interval $[t_1;t_2]$.

Proof. Let $[t_1;t_3)$ be the maximum interval of existence for $y_2(t)$. It is to show, that $t_3 \geq t_2$. Suppose $t_3 < t_2$. Then taking into account the inequality $y_2(t_1) \geq y_0(t_1)$, and (2.12_0) we conclude: it follows from a_2) and b_2), that for $p_1(t;w) \equiv P(t), q_1(t;w) \equiv Q(t)$, $r_1(t;w) \equiv R(t)$ the inequality $y_2(t) \geq y_0(t)$, $t \in [t_1;t_3)$,

holds. Since $y_0(t)$ is continuous on the interval $[t_1; t_3]$, it follows from the last inequality, that the function $f(t) \equiv \int_{t_1}^{t} \frac{y_2(\tau)}{P(\tau)} d\tau$ is bounded from below on the interval $[t_1; t_3]$. It follows from here (see [9, p. 3, the lemma 2.2]), that $[t_1; t_3)$ is not maximum interval of existence for $y_2(t)$. The obtained contradiction shows, that $t_3 \geq t_2$. The proof of the lemma is complete.

Lemma 2.7. Let $r_0(t; w) \geq 0$, $t \in [T_0; T)$, $w \in (-\infty; +\infty)$ $(t_0 \leq T_0 < T \leq +\infty)$, $\phi_0(T_1) = 0$ for some $T_1 \in [T_0; T)$, and let $\sup_{t \in [T_0; T)} |\phi_0(t)| > 0$. Then $\phi_0(t)$ changes its sign on the interval $[T_0; T)$.

Proof. Since $\sup_{t \in [T_0; T)} |\phi_0(t)| > 0$, there exists $T_2 \in [T_0; T)$ such, that $\phi_0(T_2) \neq 0$. Let $T_2 < T_1$ (the proof in the case $T_2 > T_1$ by analogy). Then there exists $T_3 \in (T_2; T_1]$ such, that

$$\phi_0(T_3) = 0, \tag{2.29}$$

$$\phi_0(t) \neq 0, \qquad t \in [T_2; T_3). \tag{2.30}$$

By Lagrange's mean value theorem it follows from (2.29) the existence of a $\xi \in (T_2; T_3)$ such, that $\phi_0'(\xi) = -\frac{\phi_0(T_2)}{T_3 - T_2}$. So,

$$sign\, \phi_0'(\xi) = -sign\, \phi_0(T_2). \tag{2.31}$$

To complete the proof of the lemma it is enough to show, that

$$sign\, \phi_0'(T_3) = -sign\, \phi_0(T_2). \tag{2.32}$$

It follows from (2.30), that $y_0(t)$ exists on the interval $[T_2; T_3]$. Therefore, by virtue of (2.5) it follows from the nonnegativity of $r_0(t; w)$, that $sign\, y_0(T_3) = sign\, y_0(\xi)$. From here and (2.31) it follows (2.32). The proof of the lemma is complete.

3 Some Global Solvability and Oscillatory Criteria

Let $P(t)$, $Q(t)$ and $R(t)$ be the same functions as in the previous section.

Theorem 3.1. Let $\phi_{(0)} \neq 0$, $\frac{\phi_{(1)}}{\phi_{(0)}} \geq 0$, and let for some $\varepsilon > 0$ the following inequalities hold

$$p_0(t; w) \geq P(t), \quad \frac{q_0(t; w)}{p_0(t; w)} \geq Q(t), \quad R(t) \leq r_0(t; w) \leq 0 \tag{3.1}$$

for $|w| \leq F\big(t_0; t; \phi_{(0)}; p(t_0; \phi_{(0)}) \frac{\phi_{(1)}}{\phi_{(0)}}\big) + \varepsilon$, $t \geq t_0$. Then the solution $\phi_0(t)$ of eq. (1.1), satisfying the initial value conditions: $\phi_0(t_0) = \phi_{(0)}$, $\phi_0(t_0) = \phi_{(1)}$, exists on the interval $[t_0; +\infty)$. The function $|\phi_0(t)|$ is positive, nondecreasing and satisfies the estimate

$$|\phi_0(t)| \leq F\left(t_0; t; \phi_{(0)}; p_0(t_0; \phi_{(0)}) \frac{\phi_{(1)}}{\phi_{(0)}}\right), \qquad t \geq t_0. \tag{3.2}$$

In this case if $\phi_{(1)} \neq 0$, then

$$\phi'(t) \neq 0, \qquad t \geq t_0. \tag{3.3}$$

Proof. Let $\phi_{(0)} > 0$ (the proof in the case $\phi_{(0)} < 0$ by analogy), $\phi_0(t)$ be the solution of eq. (1.1), satisfying the initial value conditions: $\phi_0(t_0) = \phi_{(0)}$, $\phi_0'(t_0) = \phi_{(1)}$ (existence of $\phi_0(t)$ follows from the connection between (1.1) and (1.2) and from the Peano's theorem; see [3, p. 21, the theorem 2.1]), and let $[t_0; T)$ be the maximum interval of existence for $y_0(t) \equiv p_0(t; \phi_0(t)) \frac{\phi_0'(t)}{\phi_0(t)}$ (it assumes, that $\phi_0(t)$ exists on the interval $[t_0; T)$ and does not vanish on it). Let us show, that

$$y_0(t) \geq 0, \qquad t \in [t_0; T). \tag{3.4}$$

Suppose, that it is not so. Then since $y_0(t_0) = p_0(t_0; \phi_{(0)}) \frac{\phi_{(1)}}{\phi_{(0)}} \geq 0$, there exists $\widetilde{t_0}, \ \widetilde{t_1}$ such, that

$$y_0(t) \geq 0, \qquad t \in [t_0; \widetilde{t_0}], \tag{3.5}$$

$$y_0(t) < 0, \qquad t \in (\widetilde{t_0}; \widetilde{t_1}). \tag{3.6}$$

By (3.1) for $c_1 = \phi_{(0)}, \ c_2 = p_0(t_0; \phi_{(0)}) \frac{\phi_{(1)}}{\phi_{(0)}}, \ t_1 = t_0$ the inequalities (2.20) of the lemma 2.3 hold. Therefore taking into account (2.2) we will have:

$$\phi_0(t) \leq F\left(t_0; t; \phi_{(0)}; p_0(t_0; \phi_{(0)}) \frac{\phi_{(1)}}{\phi_{(0)}}\right), \qquad t \in [t_0; \widetilde{t_0}].$$

By (2.2) from here and from (3.6) it follows:

$$\widetilde{M}(t) \equiv \max_{\xi \in [t_0; t]} \phi_0(\xi) \leq F\left(t_0; t; \phi_{(0)}; p_0(t_0; \phi_{(0)}) \frac{\phi_{(1)}}{\phi_{(0)}}\right), \qquad t \in [t_0; \widetilde{t_1}).$$

Then taking into account third of the inequalities (3.1) we will have: $r_0(t; w) \leq 0$ for $0 < w \leq \widetilde{M}(t), \ t \in [t_0; \widetilde{t_1})$. On the strength of the lemma 2.1 we conclude from here, that $y_0(t) \geq 0$ for $t \in [t_0; \widetilde{t_1})$, which contradicts (3.6). The obtained contradiction proves (3.4). Let us show, that $T = +\infty$. Suppose $T < +\infty$. By virtue of the lemma 2.3 from (3.1) and (3.3) it follows:

$$\int_{t_0}^{t} \frac{y_0(\tau)}{p_0(\tau; \phi_0(\tau))} d\tau \leq y_0(t_0) I_{P,Q}^+(t_0; t) - \int_{t_0}^{t} I_{Q,R}^-(t_0; \tau) \frac{d\tau}{P(\tau)}, \qquad t \in [t_0; T). \tag{3.7}$$

Therefore, $\int_{t_0}^{T} \frac{y_0(\tau)}{p_0(\tau; \phi_0(\tau))} d\tau < +\infty$. On the strength of the lemma 2.2 it follows from here, that $[t_0; T)$ is not maximal interval of existence for $y_0(t)$. The obtained contradiction shows, that $T = +\infty$. So (3.7) is valid for all $t \geq t_0$. Therefore, by (2.2) the inequality (3.2) holds. Since $\phi_0(t_0) = \phi_{(0)} > 0$, and $y_0(t)$ is nonnegative, by (2.2) the function $\phi_0(t)$ is positive and nondecreasing on the interval $[t_0; +\infty)$. And if $\phi_{(1)} > 0$, then by (2.2) and the lemma 2.1 the inequality (3.3) holds. The proof of the theorem is complete.
Remark 3.1. A solution $\phi_*(t)$ of the equation

$$(P(t)\phi'(t))' + Q(t)\phi'(t) + R(t)\phi(t) = 0, \qquad , t \geq t_o,$$

such, that $\phi_*(t) \neq 0, \ t \in [t_1; t_2)$, is connected with the function F by the following relation

$$|\phi_*(t_1)| \exp\left\{\frac{\phi_*(t)}{\phi_*(t_1)}\right\} = F\left(t_1; t; \phi_*(t_1); P(t_1) \frac{\phi_*'(t_1)}{\phi_*(t_1)}\right) \exp\left\{1 - P(t_1) \frac{\phi_*'(t_1)}{\phi_*(t_1)} \times \right.$$

$$\left. \times \int_{t_1}^{t} \frac{d\tau}{P(\tau)} \int_{t_1}^{\tau} \exp\left\{-\int_{s}^{\tau} \frac{Q(\xi)}{P(\xi)} d\xi\right\} R(s) I_{P,Q}^+(t_1; s) ds + \frac{1}{\phi_*(t_1)}(K^2 \phi_*)(t)\right\}, \tag{3.8}$$

where K - is the integral operator

$$(K\phi_*)(t) \equiv \int_{t_1}^{t} \frac{d\tau}{P(\tau)} \int_{t_1}^{\tau} \exp\left\{-\int_{s}^{\tau} \frac{Q(\xi)}{P(\xi)} d\xi\right\} R(s) \phi_*(s) ds, \qquad , t \in [t_1; t_2).$$

Indeed, since $y_*(t) \equiv P(t) \frac{\phi_*'(t)}{\phi_*(t)}$ is a solution of eq. (2.8) on the interval $[t_1; t_2)$, by the Cauchy's formula

$$y_*(t) = y_*(t_1) \exp\left\{-\int_{t_1}^{t} \frac{Q(\tau)}{P(\tau)} d\tau - \int_{t_1}^{t} \frac{y_*(\tau)}{P(\tau)} d\tau\right\} -$$

$$- \int\limits_{t_1}^{t} \exp\left\{ -\int\limits_{\tau}^{t} \frac{Q(s)}{P(s)} \mathrm{d}s - \int\limits_{\tau}^{t} \frac{y_*(s)}{P(s)} \mathrm{d}s \right\} R(\tau) \mathrm{d}\tau, \qquad t \in [t_1; t_2)$$

Multiplying both sides of this equality on $\frac{1}{P(t)} \exp\left\{ \int\limits_{t_1}^{t} \frac{y_*(\tau)}{P(\tau)} \mathrm{d}\tau \right\}$ and integrating from t_1 to t taking into account the equality $\phi_*(t) = \phi_*(t_1) \exp\left\{ \int\limits_{t_1}^{t} \frac{y_*(\tau)}{P(\tau)} \mathrm{d}\tau \right\}, \ t \in [t_1; t_2)$, we obtain:

$$\frac{\phi_*(t)}{\phi_*(t_1)} = 1 + P(t_1) \frac{\phi'_*(t_1)}{\phi_*(t_1)} I^+_{P,Q}(t_1; t) - \frac{1}{\phi_*(t_1)} (K\phi_*)(t), \qquad t \in [t_1; t_2).$$

After making the first iteration in this equality, taking its exponential and multiplying by $|\phi_*(t_1)|$ we come to (3.8). The question of an application of the equality (3.8) for establishing effective criteria of global solvability of eq (1.1) is an issue of separate study.

Using the lemma 2.4 in place of the lemma 2.3 by analogy it can be proved

Theorem 3.2. Let $\phi_{(0)} \neq 0$, $\frac{\phi_{(1)}}{\phi_{(0)}} \geq 0$, and let for some $\varepsilon > 0$ the following inequalities hold:

$$p_0(t; w) \geq P(t), \ \ q_0(t; w) \geq Q(t) \geq 0, \ \ r_0(t; w) \leq 0, \ \ \left| p_0(t; w) \frac{r_0(t; w)}{q_0(t; w)} \right| \leq \widetilde{Q}(t)$$

for $|w| \leq G_M\left(t_0; t; \phi_{(0)}; p_0(t_0; \phi_{(0)}) \frac{\phi_{(1)}}{\phi_{(0)}} \right) + \varepsilon, \ t \geq t_0$. Then the solution $\phi_0(t)$ of eq. (1.1), satisfying the initial value conditions $\phi_0(t_0) = \phi_{(0)}, \ \phi'(t_0) = \phi_{(1)}$, exists on the interval $[t_0; +\infty)$. The function $|\phi_0(t)|$ is positive, nondecreasing and satisfies the inequality

$$|\phi_0(t)| \leq G_M\left(t_0; t; \phi_{(0)}; p_0(t_0; \phi_{(0)}) \frac{\phi_{(1)}}{\phi_{(0)}} \right), \ \ t \geq t_0.$$

And if $\phi_{(1)} \neq 0$, then $\phi'_0(t) \neq 0, \ t \geq t_0$.

Theorem 3.3. Let $\phi_1(t)$ be a solution of eq. (2.8) on the interval $[t_0; +\infty)$, and $\phi_0(t)$ be a solution of eq. (1.1) such, that

A_1) $\phi_1(t_0) \geq \phi_0(t_0) > 0, \ \phi'_1(t_0) > \phi_0(t_0) \geq 0$ or
$\quad \phi_1(t_0) \leq \phi_0(t_0) < 0, \ \phi'_1(t_0) < \phi_0(t_0) \leq 0$;
B_1) $p_0(t_0; \phi_0(t_0)) \frac{\phi'_0(t_0)}{\phi_0(t_0)} < p_1(t_0; \phi_1(t_0)) \frac{\phi'_1(t_0)}{\phi_1(t_0)}$.
Let in addition the following conditions hold:
C_1) $p_0(t; w) \equiv p_1(t; w)$ is a non increasing by w on the interval $(-\infty; 0]$ and non decreasing by w on the interval $[0; +\infty)$ function;
D_1) $r_1(t; w_1) \leq r_0(t; w) \leq 0$ for $t \geq t_0, \ |w| \leq |w_1|, \ w, \ w_1 \in (-\infty; +\infty)$;
E_1) $\frac{q_0(t; w)}{p_0(t; w)} \leq \frac{q_1(t; w_1)}{p_1(t; w_1)}$ for $t \geq t_0, \ |w| \leq |w_1|, \ w, \ w_1 \in (-\infty; +\infty)$.
Then $\phi_0(t)$ exists on the interval $[t_0; +\infty)$, and the function $|\phi_0(t)|$ is positive and non decreasing.

Proof. Let $[t_0; T)$ be the maximum interval of existence for $\phi_0(t)$. Let us show, that

$$\phi_0(t) \neq 0, \qquad t \in [t_0; T). \tag{3.9}$$

Suppose, that this relation is false. Then it follows from A_1), that for some $T_1 \in (t_0; T)$

$$\phi_0(t) \neq 0, \qquad t \in [t_0; T_1); \tag{3.10}$$

$$\phi_0(T_1) = 0. \tag{3.11}$$

It follows from (3.10), that $y_0(t) \equiv p_0(t; \phi_0(t)) \frac{\phi'_0(t)}{\phi_0(t)}$ exists at least on the interval $[t_0; T_1)$. It follows from A_1), that $y_0(t) \geq 0, \ t \in [t_0; T_1)$. By (2.2) from here and A_1) it follows: $\phi_0(T_1) \neq 0$, which contradicts (3.11). The obtained contradiction proves (3.9). It follows from (3.9), that $y_0(t)$ exists on the interval $[t_0; T)$. Then since $y_0(t_0) \geq 0$, by virtue of (2.5) it follows from D_1), that

$$y_0(t) \geq 0, \qquad t \in [t_0; T). \tag{3.12}$$

Since by condition of the theorem $\phi_1(t) \neq 0$, $t \geq t_0$, $y_1(t) \equiv p_1(t; \phi_1(t)) \frac{\phi_1'(t)}{\phi_1(t)}$ exists on the interval $[t_0; +\infty)$. It follows from B_1), that $y_1(t_0) > y_0(t_0)$. On the strength of lemma 2.5 it follows from here, A_1) and C_1) - E_1), that

$$y_0(t) < y_1(t), \qquad t \in [t_0; T). \tag{3.13}$$

Let us show, that $T = +\infty$. Suppose $T < +\infty$. Then from B_1) and (3.13) it follows:

$$\int\limits_{t_0}^{T} \frac{y_0(\tau)}{p_0(\tau; \phi_0(\tau))} \mathrm{d}\tau \leq \int\limits_{t_0}^{T} \frac{y_1(\tau)}{p_0(\tau; 0)} \mathrm{d}\tau < +\infty.$$

Using lemma 2.2 from here we conclude, that $[t_0; T)$ is not maximum interval of existence for $y_0(t)$. But in the other hand since $[t_0; T)$ is the maximum interval of existence for $\phi_0(t)$, then $[t_0; T)$ is the maximum interval of existence for $y_0(t)$. We came to the contradiction. The obtained contradiction shows, that $T = +\infty$. Thus, $\phi_0(t)$ exists on the interval $[t_0; +\infty)$. Due to (2.2) it follows from A_1) and (3.12), that the function $|\phi_0(t)|$ is positive and nondecreasing. The proof of the theorem is complete.

Definition 3.1. We call a solution $\phi_0(t)$ of eq. (1.1) singular oscillatory of second kind, if the existence domain of the $\phi_0(t)$ is a bounded set, and if $\phi_0(t)$ infinitely many times changes its sign.

Theorem 3.4. Let the following conditions hold:

A_2) $p_0(t; w) \geq P(t)$, $\frac{q_0(t;w)}{p_0(t;w)} \geq Q(t)$, $t \geq t_0$, $w \in (-\infty; +\infty)$;

B_2) $r_0(t; w) \geq 0$, $t \geq t_0$, $w \in (-\infty; +\infty)$,

Then for each $\phi_{(0)}$ and $\phi_{(1)}$ a non-extendable on the interval $[t_0; +\infty)$ solution $\phi_0(t)$ of eq. (1.1), satisfying the initial value conditions $\phi_0(t_0) = \phi_{(0)}$, $\phi_0'(t_0) = \phi_{(1)}$, is singular oscillatory of second kind.

Proof. Let $[t_0; T)$ $(T < +\infty)$ be the maximum interval of existence for $\phi_0(t)$. Then, it is evident, that

$$\sup_{\xi \in [T_1; T)} |\phi_0(\xi)| > 0, \quad T_1 \in [t_0; T). \tag{3.14}$$

(otherwise $\phi_0(t)$ will be extended by zero, i. e. $\phi_0(t) \equiv 0$) Let us show, that for each $T_1 \in [t_0; T)$ the function $\phi_0(t)$ has a zero on the interval $[T_1; T)$. Suppose, that for some $T_0 \in [t_0; T)$ the function $\phi_0(t)$ has no zero on the interval $[T_0; T)$. Let then $\phi_0(t) > 0$, $t \in$ $\in [T_0; T)$ (the proof in the case $\phi_0(t) < 0$, $t \in [T_0; T)$, by analogy). By (2.6) it follows from here, from A_2) and B_2), that

$$\phi_0(t) \leq \frac{p_0(T_0; \phi(T_0))}{P(t)} |\phi'(T_0)| \exp\left\{ -\int\limits_{T_0}^{t} Q(\tau) \mathrm{d}\tau \right\}, \qquad t \in [T_0; T).$$

Therefore, $\phi_0(t)$ is bounded. Then due to (2.6) and (2.7) there exists finite limits $\lim\limits_{t \to T-0} \phi_0(t)$, $\lim\limits_{t \to T-0} \phi_0'(t)$. It follows from here, that $[t_0; T)$ is not maximum interval of existence for $\phi_0(t)$. The obtained contradiction shows, that for every $T_1 \in [t_0; T)$ the function $\phi_0(t)$ has a zero on the interval $[T_1; T)$. On the basis of lemma 2.7 we conclude that from here, from B_2) and (3.14) it follows, that $\phi_0(t)$ is a singular oscillatory solution of second kind. The proof of the theorem is complete.

Definition 3.2. We call a solution of eq. (1.1) oscillatory, if it exists on the interval $[t_0; +\infty)$ and in every neighborhood of $+\infty$ changes its sign.

Definition 3.3. We call a solution $\phi(t)$ of eq. (1.1) singular oscillatory of first kind, if it exists on the interval $[t_0; +\infty)$, $supp\, \phi(t)$ is bounded and $\phi(t)$ infinitely many times changes its sign.

Let for every $\varepsilon > 0$ the functions $p_\varepsilon(t)$, $q_\varepsilon(t)$ and $r_\varepsilon(t)$ be real valued and continuous on the interval $[t_0; +\infty)$, and let $p_\varepsilon(t) > 0$, $t \geq t_0$, $\varepsilon > 0$. Consider the family of equations.

$$(p_\varepsilon(t)\phi'(t))' + q_\varepsilon(t)\phi'(t) + r_\varepsilon(t)\phi(t) = 0, \qquad t \geq t_0, \qquad \varepsilon > 0. \tag{3.15_ε}$$

Theorem 3.5. Let the following conditions hold:

A_3) $r(t; w) \geq 0$, $t \geq t_0$, $w \in (-\infty; +\infty)$;

B$_3$) there exists $\varepsilon_0 > 0$ such, that for every $\varepsilon \in (0; \varepsilon_0]$

$$p_0(t; w) \leq p_\varepsilon(t), \quad \frac{q_0(t; w)}{p_0(t; w)} \leq \frac{q_\varepsilon(t)}{p_\varepsilon(t)}, \quad r_0(t; w) \geq r_\varepsilon(t), \quad \text{for } |w| \geq \varepsilon, \ t \geq t_0,$$

and eq. (3.15$_\varepsilon$) is oscillatory;

C$_3$) there exists $N > 0$ such, that

C$_3^1$) $p_0(t; w) \leq P(t), \ \frac{q_0(t;w)}{p_0(t;w)} \leq Q(t)$ for $|w| \leq N, \ t \geq t_0$ and

$$\int\limits_{t_0}^{+\infty} \exp\left\{ - \int\limits_{t_0}^{\tau} Q(s) \mathrm{d}s \right\} \frac{\mathrm{d}\tau}{P(\tau)} = +\infty; \tag{3.16}$$

C$_3^2$) for every $\varepsilon \geq N$ the following inequalities hold: $p_0(t; w) \leq p_\varepsilon(t), \ \frac{q_0(t;w)}{p_0(t;w)} \leq q_\varepsilon(t),$
$r_0(t; w) \geq r_\varepsilon(t)$ for $N \leq |w| \leq \varepsilon$ and

$$\int\limits_{t_0}^{+\infty} \frac{\mathrm{d}\tau}{P(\tau)} \int\limits_{t_0}^{\tau} \exp\left\{ - \int\limits_{s}^{\tau} q_\varepsilon(\xi) \mathrm{d}\xi \right\} r_\varepsilon(s) \mathrm{d}s = +\infty. \tag{3.17}$$

Then each existing on the interval $[t_0; +\infty)$ nontrivial solution of eq. (1.1) either oscillatory, or singular oscillatory of first kind.

Proof. Let $\phi_0(t)$ be a solution of eq. (1.1) such, that $supp\ \phi_0(t)$ is unbounded on the interval $[t_0; +\infty)$. Let us show, that $\phi_0(t)$ has arbitrarily large zeros. Suppose, that it is not so, i. e. there exists $t_1 \geq t_0$ such, that $\phi_0(t) \neq 0, \ t \geq t_1$. Let then $\phi_0(t) > 0, \ t \geq t_1$ (the proof in the case $\phi_0(t) < 0, \ t \geq t_1$ by analogy). Due to (2.6) it follows from A$_3$), that there can be one of the following three cases.

α) $\phi_0'(t) \geq 0, \quad t \geq t_1$;
β) there exists $t_2 \geq t_1$ such, that $\phi_0(t) \leq N, \ t \geq t_2, \ \phi_0'(t_2) < 0$;
γ) there exists $t_2 \geq t_1$ such, that $\phi_0(t) \geq N, \ t \geq t_2, \ \phi_0'(t_2) < 0$.

Let the case α) holds, and let $0 < \varepsilon < \min\{\phi_0(t_1); \varepsilon_0\}$. Then $y_0(t) \equiv p_0(t; \phi_0(t))\frac{\phi_0'(t)}{\phi_0(t)}$ is a solution of eq. (2.1) on the interval $[t_2; +\infty)$, and $y_0(t) \geq 0, \ t \geq t_1$. By virtue of lemma 2.6 it follows from here and from B$_3$), that the Riccati's equation

$$y'(t) + \frac{y^2(t)}{p_\varepsilon(t)} + \frac{q_\varepsilon(t)}{p_\varepsilon(t)} y(t) + r_\varepsilon(t) = 0, \quad t \geq t_1,$$

has a solution on the interval $[t_1; +\infty)$. Consequently, corresponding equation (3.15$_\varepsilon$) is not oscillatory, which contradicts C$_3$). The obtained contradiction shows, that $\phi_0(t)$ has arbitrary large zeroes. Let the condition β) holds. Then it follows from C$_3^1$), that

$$\int\limits_{t_2}^{t} \exp\left\{ - \int\limits_{t_2}^{\tau} \frac{q_0(s; \phi_0(s))}{p_0(s; \phi_0(s))} \right\} \frac{\mathrm{d}\tau}{p_0(\tau; \phi_0(\tau))} \geq \int\limits_{t_2}^{t} \exp\left\{ - \int\limits_{t_2}^{\tau} Q(s) \mathrm{d}s \right\} \frac{\mathrm{d}\tau}{P(\tau)}, \quad t \geq t_2 \tag{3.18}$$

It is easy to show, that from (3.16) it follows equality $\int\limits_{t_2}^{+\infty} \exp\left\{ - \int\limits_{t_2}^{\tau} Q(s) \mathrm{d}s \right\} \frac{\mathrm{d}\tau}{P(\tau)} = +\infty$. By (2.7) from here, from A$_3$) and (3.18) it follows: $\lim\limits_{t \to +\infty} \phi_0(t) = -\infty$, which contradicts the suggestion: $\phi_0(t) > 0, \ t \geq t_1$. The obtained contradiction shows, that $\phi_0(t)$ has arbitrary large zeroes. Let the case γ) holds. Then from C$_3^2$) it follows:

$$\int\limits_{t_2}^{t} \frac{\mathrm{d}\tau}{p_0(\tau; \phi_0(\tau))} \int\limits_{t_2}^{\tau} \exp\left\{ - \int\limits_{s}^{\tau} \frac{q_0(\xi; \phi_0(\xi))}{p_0(\xi; \phi_0(\xi))} \mathrm{d}\xi \right\} r_0(s; \phi_0(s)) \phi_0(s) \mathrm{d}s \geq$$

$$\geq \int\limits_{t_2}^{t} \frac{\mathrm{d}\tau}{p_\varepsilon(\tau)} \int\limits_{t_2}^{\tau} \exp\Big\{-\int\limits_{s}^{\tau} q_\varepsilon(\xi)\Big\} r_\varepsilon(\tau)\mathrm{d}\tau$$

By (2.7) from here and from (3.17) it follows: $\lim\limits_{t\to+\infty} \phi_0(t) = -\infty$, which contradicts the suggestion: $\phi_0(t) > 0$, $t \geq t_1$. The obtained contradiction shows, that $\phi_0(t)$ has arbitrary large zeroes. Thus, we showed, that a solution of eq. (1.1), existing on the interval $[t_0; +\infty)$ and *supp* of which is an unbounded set, has arbitrary large zeroes. Due to lemma 2.7 it follows from here and from A_3), that $\phi_0(t)$ is oscillatory.

Let $\phi_0(t)$ be a finite solution of eq. (1.1) on the interval $[t_0; +\infty)$, and let $\phi_0(t) = = 0$, $t \geq T$;

$$\max_{\xi\in[t_1;T]} |\phi_0(\xi)| > 0, \quad t_1 \in [t_0; T). \tag{3.19}$$

Let us show, that for each $T_1 \in [t_0; T)$ the function $\phi_0(t)$ has a zero on the interval $[T_1; T)$. Suppose, that it is not so, i .e. there exists $T_0 \in [t_0; T)$ such, that $\phi_0(t)$ does not vanish on the interval $[T_0; T)$. Since $\phi_0(T) = 0$, on the strength of Lagrange's mean value theorem there exists $\xi \in [T_0; T)$ such, that $\phi_0'(\xi) \neq 0$, and $sign\ \phi_0(T_0) = -sign\ \phi_0'(\xi)$. By (2.6) it follows from here and from A_3), that $\phi_0'(T) \neq 0$. But, in the other hand, since $\phi_0(t) = 0$ for $t \geq T$, we have $\phi_0'(T) = 0$. We came to the contradiction. Consequently, for every $T_1 \in [t_0; T)$ the function $\phi_0(t)$ has a zero on the interval $[T_1; T)$. Due to lemma 2.7 it follows from here, from A_3) and (3.19), that $\phi_0(t)$ is a singular oscillatory solution of first kind for eq. (1.1). The proof of the theorem is complete.

Remark 3.2. If the solution $\phi_0(t)$ of eq. (1.1), satisfying the initial value conditions $\phi_0(t_0) = \phi_0'(t_0) = 0$ is unique, $(\phi_0(t) \equiv 0)$, in particular, if $p_0(t; w)$, $q_0(t; w)$ and $r_0(t; w)$ satisfy the conditions of the remark 1.1, then eq. (1.1) has no singular oscillatory solutions of first kind.

Theorem 3.6. Let the conditions hold:

A_4) $r_0(t; w) \geq 0$ for $t \geq t_0$, $w \in (-\infty; +\infty)$;

B_4) $p_0(t; w), \frac{q_0(t;w)}{p_0(t;w)}, \quad -r_0(t; w)$ are non increasing by w on the interval $(-\infty; 0]$ and nondecreasing by w on the interval $[0; +\infty)$ functions.

Then for each $\phi_{(0)}$ and $\phi_{(1)}$ the solution $\phi_0(t)$ of eq. (1.1), satisfying the initial value conditions:

$$\phi_0(t_0) = \phi_{(0)}, \quad \phi_0'(t_0) = \phi_{(1)}, \tag{3.20}$$

exists on the interval $[t_0; +\infty)$.

Proof. Let $\phi_0(t)$ be a solution of eq. (1.1), satisfying the initial value conditions (3.19), and let $[t_0; T)$ be the maximum interval of existence for $\phi_0(t)$. We should show, that $T = +\infty$. Suppose $T < +\infty$. Two cases are possible:

α) there exists $t_1 \in [t_0; T)$ such, that $\phi_0(t) \neq 0$, $t \in [t_1; T)$;

β) there exists infinite sequence $t_0 < t_0 < t_2 < ... < T$ such, that $\lim\limits_{k\to+\infty} t_k = T$, $\phi_0(t_k) = 0$, $k = 1, 2, ...$

Let the case α) holds, and let $\phi_0(t) > 0$, $t \in [t_1; T)$ (the proof in the case $\phi_0(t) < 0$, $t \in [t_1; T)$, by analogy). If $\phi_0'(t) \leq 0$, then by virtue of (2.7) it follows from A_4), that $\phi_0(t)$ has finite (nonnegative) limit when $t \to T - 0$. Then by virtue of (2.6) the function $\phi_0'(t)$ also has finite limit when $t \to T - 0$. Therefore, $\phi_0(t)$ continues to right at T,, so $[t_0; T)$ is not maximum interval of existence for $\phi_0(t)$. The obtained contradiction shows, that $T = +\infty$. Let $\phi_0'(t_1) > 0$. Two subcases are possible:

α_1) $\phi_0'(t) \geq 0$, $t \in [t_1; T)$;

β_1) $\phi_0'(t) \geq 0$, for $t \in [t_1; T_1]$, $\phi_0'(t) < 0$, for $t \in (T_1; T)$, for some $T_1 \in (t_1; T)$ (by virtue of (2,2) and (2.5) it follows from A_4), that if $\phi_0'(\tilde{t}_1) < 0$ for some $\tilde{t}_1 \in (t_1; T]$, then $\phi_0'(t) < 0$, $t \in [\tilde{t}_1; T]$).

In the case α_1) the function $\phi_0(t)$ is nondecreasing on the interval $(t_1; T)$. Then (2.6) it follows from A_4) and B_4), that

$$\phi_0'(t) \leq \phi_0'(t_1) \exp\Big\{-\int\limits_{t_1}^{t} \frac{q_0(\tau; \phi_0(t_1))}{p_0(\tau; \phi_0(t_1))}\mathrm{d}\tau\Big\}, \quad t \in [t_1; T).$$

Therefore, $\phi_0(t)$ has a finite limit when $t \to T - 0$. It is evident, that the same we have in the case β_1). Then by (2.6) $\phi_0'(t)$ has finite limit when $t \to T - 0$. So, $[t_0; T)$ is not maximum interval of existence for $\phi_0(t)$. The obtained contradiction shows, that $T = +\infty$. Let the case β) takes place. If $\phi_0(t)$ is bounded,

then, it is evident, that the functions $\frac{1}{p_0(t,\phi_0(t))}$, $\frac{q_0(t,\phi_0(t))}{p_0(t,\phi_0(t))}$, $r_0(t,\phi_0(t))$ are bounded in the interval $[t_0;T]$. By virtue of (2.7) it follows from here, that $\phi_0(t)$ has finite limit when $t \to T - 0$. Then arguing similarly to the above (when we are dealing with the cases α_1) and β_1)) we conclude, that $T = +\infty$. Let $\phi_0(t)$ be not bounded. Two subcases are possible:

α_2) $\varlimsup\limits_{t \to T-0} \phi_0(t) = +\infty$;

β_2) $\varliminf\limits_{t \to T-0} \phi_0(t) = -\infty$.

Let the case α_2) holds (the proof in the case β_2) by analogy). Let $\phi_1(t)$ be the solution of eq. (1.1) with $\phi_1(T) = 1$, $\phi_1'(T) = -1$. Then there exists $\zeta \in [t_1; T)$ such, that $\phi_1(t)$ exists on the interval $[\zeta; T]$, and
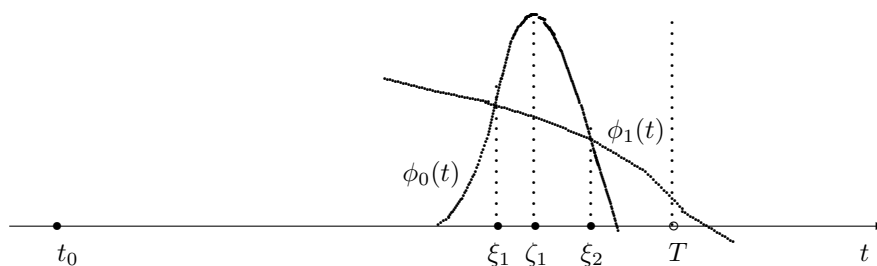
$$\phi_1(t) > 0, \quad \phi_1'(t) < 0, \qquad t \in [\zeta; T]. \tag{3.21}$$

It follows from β) and α_2), that there exists a point $\zeta_1 (\in [\zeta; T))$ of local maximum for $\phi_0(t)$ such, that $\phi_0(\zeta_1) > \max\limits_{\xi \in [\zeta;T]} \phi_1(\xi)$ and $\zeta < t_m < \zeta_1$ for some m. Since $t_m < \zeta_1 < t_n$ for some $n > m$ and $\phi_0(t_m) = \phi_0(t_n) = 0 < \min\{\phi_1(t_m), \phi_1(t_n)\}$, but $\phi_0(\zeta_1) > \max\{\phi_1(t_m), \phi_1(t_n)\}$, there exist ξ_1 and ξ_2 such, that $\phi_0(\xi_k) = \phi_1(\xi_k)$, $k = 1, 2$ and

$$\phi_0(t) > \phi_1(t), \qquad t \in (\xi_1; \xi_2). \tag{3.22}$$

(see pict. 1). Let $y_j(t) \equiv p_0(t; \phi_j(t))\frac{\phi_j'(t)}{\phi_j(t)}$, $j = 0, 1$. Then by virtue of (2.12_1) the following equality holds:

$$y_1(t) - y_0(t) = [y_1(\zeta_1) - y_0(\zeta_1)] \exp\left\{-\int\limits_{\zeta_1}^{t} \frac{y_0(\tau) + y_1(\tau) + q_0(\tau; \phi_0(\tau))}{p_0(\tau; \phi_0(\tau))} d\tau\right\} -$$

$$-\int\limits_{\zeta_1}^{t} \exp\left\{-\int\limits_{\tau}^{t} \frac{y_0(s) + y_1(s) + q_0(s; \phi_0(s))}{p_0(s; \phi_0(s))} ds\right\}\left[\left(\frac{1}{p_1(\tau; \phi_1(\tau))} - \frac{1}{p_0(\tau; \phi_0(\tau))}\right)y_1^2(\tau) + \right.$$

$$\left. + \left(\frac{q_1(\tau; \phi_1(\tau))}{p_1(\tau; \phi_1(\tau))} - \frac{q_0(\tau; \phi_0(\tau))}{p_0(\tau; \phi_0(\tau))}\right)y_1(\tau) + r_1(\tau; \phi_1(\tau)) - r_0(\tau; \phi_0(\tau))\right]d\tau = 0. \tag{3.23}$$

pict. 1

An illustration to the part of proof of Theorem 3.6, related to the case α_2). Here $[t_0; T)$ is the maximum existence interval for $\phi_0(t)$. The "cap" in this picture is a part of the graph of $\phi_0(t)$. The decreasing curve is a part of the graph of $\phi_1(t)$; ξ_1 and ξ_2 are points of intersection of graph of $\phi_0(t)$ and ϕ_1; ζ_1 is a local maximum point of $\phi_0(t)$.

It is evident, that

$$y_1(\zeta_1) < y_0(\zeta_1) \ (= 0). \tag{3.24}$$

Since $\phi_0(t) \geq \phi_1(t) > 0$ on the interval $[\zeta_1; \xi_2]$, then it follows from the conditions of the theorem, that

$$p_0(t; \phi_0(t)) \geq p_0(t; \phi_1(t)), \ r_0(t; \phi_0(t)) \leq r_0(t; \phi_1(t)), \ \frac{q_0(t; \phi_0(t))}{p_0(t; \phi_0(t))} \geq \frac{q_0(t; \phi_1(t))}{p_0(t; \phi_1(t))}, \ t \in [\zeta_1; \xi_2].$$

Then since $y_1(t) < 0$ on the interval $[\zeta_1; \xi_2]$, it follows from (3.23) and (3.24), that $y_1(\xi_2) < y_0(\xi_2)$. Then $\phi_1'(\xi_2) < \phi_0'(\xi_2)$. It follows from here, that $\phi_1'(t) < \phi_0'(t)$, $t \in [\xi_3; \xi_2]$, for some $\xi_3 \in (\zeta_1; \xi_2)$. Therefore, $\int_{\xi_3}^{\xi_2}(\phi_1(\tau) - \phi_0(\tau))'d\tau < 0$, or, which is the same, $\phi_1(\xi_3) > \phi_0(\xi_3)$ (since $\phi_1(\xi_2) = \phi_0(\xi_2)$, *see pict.*1), which contradicts (3.22). The obtained contradiction shows, that $T = +\infty$. The proof of the theorem is complete.

4   Some Applications

Consider the Emden - Fowler equation (see [2], p. 171):

$$(t^\rho \phi'(t))' - t^\sigma \phi^n(t) = 0, \ \ t \geq t_0 > 0, \ \ n > 1, \ \ \rho, \ \ \sigma \in (-\infty; +\infty). \tag{4.1}$$

Along with this equation, consider the equation

$$(t^\rho \phi'(t))' - t^\sigma |\phi(t)|^{n-1}\phi(t) = 0, \qquad t \geq t_0. \tag{4.2}$$

Here $p_0(t; w) \equiv t^\rho$, $q_0(t; w) \equiv 0$, $r_0(t; w) \equiv -t^\sigma |w|^{n-1}$. We put: $P(t) \equiv t^\rho$, $Q(t) \equiv \equiv 0$, $R(t) \equiv -t^\sigma$. Let $\rho > 1$. Then

$$F(t_0; t; c_1; c_2) \leq |c_1| \exp\left\{ \frac{c_2}{\rho - 1}(t_0^{1-\rho} - t^{1-\rho}) + \frac{t^{\sigma+2-\rho} - t_0^{\sigma+2-\rho}}{(\sigma + 1)(\sigma + 2 - \rho)} - \frac{t_0^{\sigma+2-\rho}}{(\sigma + 1)(\rho - 1)} \right\}.$$

It follows from this, that if $-1 < \sigma < \rho - 1$, then

$$F(t_0; t; c_1; c_2) \leq |c_1| \exp\left\{ \frac{c_2}{\rho - 1}t_0^{1-\rho} - \frac{t_0^{\sigma+2-\rho}}{(\sigma + 1)(\sigma + 2 - \rho)} \right\} \overset{def}{=} A(t_0; c_1; c_2),$$

and if $\sigma < -1$, then

$$F(t_0; t; c_1; c_2) \leq |c_1| \exp\left\{ \frac{c_2}{\rho - 1}t_0^{1-\rho} - \frac{t_0^{\sigma+2-\rho}}{(\sigma + 1)(\rho - 1)} \right\} \overset{def}{=} B(t_0; c_1; c_2).$$

It is not difficult to see, that if

$$-1 < \sigma < \rho - 2, \qquad A\left(t_0; \phi_{(0)}; t_0^\rho \frac{\phi_{(1)}}{\phi_{(0)}}\right) < 1, \tag{4.3}$$

or

$$\sigma < -1, \qquad B\left(t_0; \phi_{(0)}; t_0^\rho \frac{\phi_{(1)}}{\phi_{(0)}}\right) < 1, \tag{4.4}$$

then for eq. (4.2) the conditions of the theorem 3.1 hold. Therefore, for every $\phi_{(0)}$, $\phi_{(1)}$, satisfying the conditions $\phi_{(0)} \neq 0$, $\frac{\phi_{(1)}}{\phi_{(0)}} \geq 0$ and one of the conditions (4.3) or (4.4), the solution $\phi_0(t)$ of eq. (4.2) with $\phi_0(t_0) = \phi_{(0)}$, $\phi_0'(t_0) = \phi_{(1)}$, exists on the interval $[t_0; +\infty)$, and if takes place (4.3), then

$$|\phi_0(t)| \leq A\left(t_0; \phi_{(0)}; t_0^\rho \frac{\phi_{(1)}}{\phi_{(0)}}\right), \qquad t \geq t_0, \tag{4.5}$$

and if takes place (4.4), then

$$|\phi_0(t)| \leq B\left(t_0; \phi_{(0)}; t_0^\rho \frac{\phi_{(1)}}{\phi_{(0)}}\right), \qquad t \geq t_0; \tag{4.6}$$

the function $|\phi_0(t)|$ is positive and nondecreasing. It is evident, that if $\phi_{(0)} > 0$, then $\phi_0(t)$ is a solution of eq. (4.1) too.

Remark 4.1. If $\phi(t)$ is a positive (negative) solution of eq. (4.2) (and $n = \frac{n_1}{n_2}$, where n_1 and n_2 are odd), then $\phi(t)$ is a solution of eq. (4.1) too. Note that equations (4.1) and (4.2) are not equivalent for all n, e. g. the equation $\phi''(t) - \frac{1}{t}\phi^2(t) = 0$ is not equivalent to equation $\phi''(t) - \frac{1}{t}|\phi(t)|\phi(t) = 0$, since the function $\phi_0(t) \equiv -\frac{2}{t}$ is a solution of the last equation on the interval $[1; +\infty)$, but not of the first one.

Let $\rho = 0$, $\quad \sigma + n + 1 < 0$. Then (see [2], p. 173) the function $\phi_B(t) \equiv \left[\frac{(\sigma+2)(\sigma+n+1)}{(n-1)^2} \right]^{\frac{1}{n-1}} t^{-\frac{\sigma+2}{n-1}}$ is a solution of eq. (4.2). It is not difficult to see, that for $p(t;w) = p_1(t;w) \equiv 1$, $q(t;w) = q_1(t;w) \equiv 0$, $r(t;w) = r_1(t;w) = -t^\sigma |w|^{n-1}$, $t \geq t_0, w \in (-\infty; +\infty)$, $\rho = 0$, $\sigma + n + 1 < 0$ for equations (2.8)and (4.2) the conditions C_1) - E_1) of the theorem 3.3 hold. Therefore, the solution $\phi_0(t)$ of eq. (4.2) with $\phi_0(t_0) \neq 0$, $\frac{\phi_0'(t_0)}{\phi_0(t_0)} \leq \frac{\phi_B'(t_0)}{\phi_B(t_0)}$, exists on the interval $[t_0; +\infty)$, and $|\phi_0(t)|$ is positive and nondecreasing. It is evident that if $\phi_0(t_0) > 0$, then $\phi_0(t)$ is a solution of eq. (4.1). For $\phi_0(t_0) < 0$ the function $\phi_0(t)$ will be a solution of eq. (4.1) if $n = \frac{n_1}{n_2}$, where n_1 and n_2 are odd. If $\rho \neq 1$, then by invertible transformation

$$\begin{cases} s = (\rho - 1)^{-1} t^{\rho-1}, \quad \phi = (\rho - 1)^{(\rho-\sigma-2)/[(\rho-1)(n-1)]} \frac{\psi}{s} \quad for \rho > 1; \\[2mm] s = (1 - \rho)^{-1} t^{1-\rho}, \quad \phi = (1 - \rho)^{-(\sigma+\rho)/[(n-1)(1-\rho)]} \psi, \quad for \rho < 1. \end{cases}$$

eq. (4.2) converts in (see [2], pp. 171, 172)

$$\psi''(s) - s^{\sigma_1} |\psi(s)|^{n-1} \psi(s) = 0, \quad s \geq s_0 > 0, \tag{4.7}$$

where

$$\sigma_1 = \begin{cases} \frac{\sigma+\rho}{\rho-1} - (n+3), \quad \rho > 1; \\[3mm] \frac{\sigma+\rho}{1-\rho}, \quad \rho < 1. \end{cases} \tag{4.8}$$

This transformation establishes a one-to-one correspondence between solutions of equations (4.2) and (4.7), and existing on the interval $[t_0; +\infty)$ solutions of eq. (4.2) transform in solutions of eq. (4.7), existing on the interval $[s_0; +\infty)$. By already proved above for $\sigma_1 + n + 1 < 0$ eq. (4.7) has two - parameter family of solutions on the interval $[s_0; +\infty)$. Therefore from (4.8) it follows, that in the cases $\rho > \max\{1, \sigma + 2\}$ and $\frac{\sigma-1}{n} + 1 < \rho < 1$ eq. (4.2) has two - parameter family of solutions on the interval $[t_0; +\infty)$.

Definition 4.1. We say, that a solution $(\phi_0(t), \psi_0(t))$ of eq. (1.2) is conditionally stable for $t \to +\infty$, if there exists one dimensional manifold $S \ni (\phi_0(t_0), \psi_0(t_0))$ such, that for every $\varepsilon > 0$ and for every solution $(\phi(t), \psi(t))$ of the system (1.2) there exists $\delta > 0$ such, that $|\phi(t) - \phi_0(t)| + |\psi(t) - \psi_0(t)| < \varepsilon$ for $t \geq t_0$, as soon as $(\phi(t_0), \psi(t_0)) \in S$ and $|\phi(t_0) - \phi_0(t_0)| + |\psi(t_0) - \psi_0(t_0)| < \delta$ (see [10], p. 314).

Definition 4.2. We say, that a solution $\phi_0(t)$ of eq. (1.1) is conditionally stable for $t \to +\infty$, if the corresponding solution $(\phi_0(t), p_0(t; \phi_0(t))\phi_0'(t))$ of the system (1.2) is conditionally stable for $t \to +\infty$.

Let us show, that if $\rho > 1$, $\sigma < -1$, then the solution $\phi_0(t) \equiv 0$ of eq. (4.1) is conditionally stable for $t \to +\infty$. Let $S = \left\{ (\phi_{(0)}, \phi_{(1)}) : 0 \leq \phi_{(0)} \leq \exp\left\{ \frac{t_0^{\sigma+2-\rho}}{(\sigma+1)(\rho-1)} \right\}, \phi_{(1)} = 0 \right\}$, and let $\phi(t)$ be a solution of eq. (4.1) with $(\phi(t_0), t_0^\rho \phi_0'(t_0)) \in S$, $\phi(t_0) \neq 0$. Then $B(t_0; \phi(t_0); 0) < 1$. By virtue of theorem 3.1 it follows from here and from (4.4), that $\phi(t)$ exists on the interval $[t_0; +\infty)$ and satisfies the inequality

$$|\phi(t)| \leq \phi(t_0) \exp\left\{ -\frac{t_0^{\sigma+2-\rho}}{(\sigma+1)(\rho-1)} \right\}, \quad t \geq t_0 > 0. \tag{4.9}$$

By virtue of (4.1) it follows from her, that

$$|t^\rho \phi'(t)| \leq \phi(t_0) \exp\left\{ -\frac{t_0^{\sigma+2-\rho}}{(\sigma+1)(\rho-1)} \right\} \left(\frac{-t_0^{\sigma+1}}{\sigma+1} \right), \quad t \geq t_0 > 0. \tag{4.10}$$

Let $\varepsilon > 0$ be fixed. We put:

$$\delta = \delta(\varepsilon) \equiv \frac{\varepsilon}{2} \left(1 - \frac{t_0^{\sigma+1}}{\sigma+1} \right)^{-1} \exp\left\{ \frac{t_0^{\sigma+2-\rho}}{(\sigma+1)(\rho-1)} \right\}$$

Let $|\phi(t_0)| + |t_0^\rho \phi'(t_0)| < \delta$. Then it follows from (4.9) and (4.10), that

$$|\phi(t)| + |t_0^\rho \phi'(t)| < \varepsilon \ \text{ for } \ t \geq t_0. \tag{4.11}$$

If $\phi(t_0) = 0$, then by virtue of remark 1.1 $\phi(t) \equiv 0$, and, therefore, in this case the relation (4.11) also takes place. Consequently, the solution $(\phi_0(t), \psi_0(t)) \equiv (0, 0)$ of the system

$$\begin{cases} \phi'(t) = \frac{\psi(t)}{t^\rho}; \\ \\ \psi'(t) = t^\sigma \phi^n(t) \end{cases}$$

is conditionally stable for $t \to \infty$. Then the solution $\phi_0(t) \equiv 0$ of eq. (4.1) is conditionally stable for $t \to \infty$. Taking into account remark 4.1 we summarize the obtained result in the following form.

Theorem 4.1. The following assertions hold.

I). Let $\rho > 1$, and let $\phi_{(0)}$ and $\phi_{(1)}$ satisfy the conditions: $\phi_{(0)} \neq 0$, $\frac{\phi_{(1)}}{\phi_{(0)}} \geq 0$ and one of the conditions (4.3), (4.4). Then the solution $\phi_0(t)$ of eq. (4.2), satisfying the initial value conditions: $\phi_0(t_0) = \phi_{(0)}$, $\phi_0'(t_0) = \phi_{(1)}$, exists on the interval $[t_0; +\infty)$. The function $|\phi_0(t)|$ is positive and nondecreasing, and if (4.3) holds, then the estimate (4.5) is valid, and if (4.4) holds, then the estimate (4.6) is valid. If $\phi_{(0)} > 0$ or if $\phi_{(0)} < 0$, $n = \frac{n_1}{n_2}$, where n_1 and n_2 are odd, then $\phi_0(t)$ is a solution of (4.1).

II). Let $\rho = 0$, $\sigma + n + 1 < 0$. Then the solution $\phi_0(t)$ of eq. (4.2) with $\phi_0(t_0) \neq 0$, $0 \leq \frac{\phi_0'(t_0)}{\phi_0(t_0)} < \frac{\phi_B'(t_0)}{\phi_B(t_0)}$, exists on the interval $[t_0; +\infty)$, and $|\phi_0(t)|$ is positive and nondecreasing. If $\phi_0(t_0) > 0$ or if $\phi_0(t_0) < 0$, $n = \frac{n_1}{n_2}$, where n_1 and n_2 are odd, then $\phi_0(t)$ is a solution of eq. (4.1).

III). In the cases $\rho > \max\{1, \sigma + 2\}$ and $\frac{\sigma - 1}{n} + 1 < \rho < 1$ eq. (4.1) has two - parameter family of solutions on the interval $[t_0; +\infty)$.

IV). If $\rho > 1$ and $\sigma < -1$, then $\phi_0(t) \equiv 0$ of eq. (4.1) is conditionally stable for $t \to +\infty$.

Remark 4.2. In the case $\rho = 0$ the existence of global solutions of eq. (4.1), which are different by their properties from described in assertion II of theorem 4.1 (the Kneser's solutions) follows from theorem 16.1 of book [1] (see [1], p. 371).

Let us compare theorem 4.1 with the following result (see [11], p. 8).

Theorem*. The following assertions hold:

i). There exists $\varepsilon > 0$ such, that every solution of eq. (4.7) with Cauchy initial conditions $|\psi(s)| \leq \varepsilon$, $|\psi'(s| \leq \varepsilon$ exists on the interval $[s_0; +\infty)$ if and only if $\sigma_1 < -n - 1$.

ii). If $\sigma_1 \geq -n - 1$, then every solution $\psi(s)$ of eq. (4.7), satisfying $\psi(\tau)\psi'(\tau) > 0$ at some $\tau \geq s_0$ is non continuable on the interval $[s_0; +\infty)$.

In the assertions I) and II) of the theorem 4.1 the region of the initial values $\phi(t_0)$, $\phi'(t_0)$ $(\psi(s_0), \psi'(s_0))$ for which the solution $\phi(t)$ $(\psi(s))$ of eq. (4.2) (of eq. (4.7)) exists on the interval $[t_0; +\infty)$ $([s_0; +\infty))$ is describes by well - defined relationships, whereas from the assertion i) of the theorem* we can not see exactly for which initial conditions $\psi(s_0)$, $\psi'(s_0)$ (except the trivial case $\psi(s_0) = \psi'(s_0) = 0$) the solution $\psi(s)$ of eq. (4.7) exists on the interval $[s_0; +\infty)$. From the assertion ii) of the theorem* it follows, that in the assertion II) of the theorem 4.1 the condition $\sigma + n + 1 < 0$ can not be replaced by weaker condition $\sigma + n + 1 \leq 0$. In this sense theorem 3.3 (which implies II)) is sharp. From the assertion ii) of theorem* and from (4.8) it follows, that in the assertion III) of theorem 4.1 the condition $\rho > \max\{1, \sigma + 2\}$ $(\frac{\sigma - 1}{n} + 1 < 1)$ can not be replaced by weaker condition $\rho \geq \max\{1, \sigma + 2\}$ $(\frac{\sigma - 1}{n} + 1 \leq 1)$. In this sense theorem 3.1 (which is used in proof of III)) is sharp.

Let $\lambda(t)$, $\mu(t)$ and $\nu(t)$ be continuous functions on the interval $[t_0; +\infty)$ and let $\lambda(t) > 0$, $\mu(t) \geq 0$, $\nu(t) \geq 0$, $t \geq t_0$. Consider the following Van der Pol's type equation (see [12]).

$$(\lambda(t)\phi'(t))' + \mu(t)(\phi^2(t) - 1)\phi'(t) + \nu(t)\phi(t) = 0, \qquad t \geq t_0. \tag{4.12}$$

Here $p_0(t; w) \equiv \lambda(t)$, $q_0(t; w) \equiv \mu(t)(w^2 - 1)$, $r_0(t; w) \equiv \nu(t)$ satisfy all of the conditions of theorem 3.6. Therefore, for each $\phi_{(0)}$ and $\phi_{(1)}$ the solution $\phi_0(t)$ of eq. (4.12), satisfying the initial value conditions: $\phi_0(t_0) = \phi_{(0)}$, $\phi_0'(t_0) = \phi_{(1)}$, exists on the interval $[t_0; +\infty)$. We put:

$$P(t) = p_\varepsilon(t) \equiv \lambda(t), \ \ Q(t) \equiv 0, \ \ q_\varepsilon(t) = \mu(t)(\varepsilon^2 - 1), \ \ r_\varepsilon(t) = \nu(t), \tag{4.13}$$

$t \geq t_0, \; \varepsilon > 0, \; N = 1$. It is not difficult to check, that if the following conditions hold:

a°). $\displaystyle\int_{t_0}^{+\infty} \frac{\mathrm{d}\tau}{\lambda(\tau)} = \int_{t_0}^{+\infty} \frac{\mathrm{d}\tau}{\lambda(\tau)} \int_{t_0}^{\tau} \exp\left\{-\int_{s}^{\tau} \frac{\mu(\xi)(\varepsilon^2 - 1)}{\lambda(\xi)}\right\} \nu(s)\mathrm{d}s = +\infty \;$ when $\; \varepsilon \geq N$;

b°). for $\; 0 < \varepsilon \leq \varepsilon_0 \;$ the equations

$$(\lambda(t)\phi'(t))' + \mu(t)(\varepsilon^2 - 1)\phi'(t) + \nu(t)\phi(t) = 0, \qquad t \geq t_0$$

are oscillatory, then for eq. (4.12) with (4.13) the conditions A_3) - B_3) of theorem 3.5 hold. Then due to theorem 3.5 the solution $\phi_0(t)$ either is oscillatory or is singular oscillatory of first kind. Since it is evident that the functions $p_0(t; w) \equiv \lambda(t), \; q_0(t; w) \equiv \mu(t)(w^2 - 1), \; r_0(t; w) \equiv \nu(t)$ satisfy the conditions of the remark 1.1, the solution $\phi_0(t)$, satisfying the initial value conditions $\phi_0(t_0) = \phi_{(0)}, \; \phi_0'(t_0) = \phi_{(1)}$, is exactly one, and consequently, due to remark 3.2 cannot be singular oscillatory of first kind. The obtained result we summarize in the following form.

Theorem 4.2. Let $\lambda(t) > 0, \; \mu(t) \geq 0, \; \nu(t) \geq 0$ for $t \geq t_0$. Then for each $\phi_{(0)}$ and $\phi_{(1)}$ the solution $\phi_0(t)$ of eq. (4.12), satisfying the initial value conditions:
$\phi_0(t_0) = \phi_{(0)}, \; \phi_0'(t_0) = \phi_{(1)}$, exists on the interval $[t_0; +\infty)$. Moreover if in addition the conditions a°) and b°) hold, then $\phi_0(t)$ is oscillatory.

5 Conclusion

We have used the Riccati equation method to investigate some classes of second order nonlinear ordinary differential equations. This method have made possible us to establish four new global existence criteria for the mentioned classes of equations. We have proved two new oscillatory criteria for them as well. These criteria were used to the Emden - Fowler equation, having applications in the astrophysics, and to the Wan der Pole type equation, which is applicable for studying the dynamics of dusty grain charge in dusty plasmas.

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