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Common Fixed Point Results in Complex Valued Metric Spaces with $(E.A)$ and (CLR) Properties

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Abstract In this article, we establish common fixed point theorems for two and four self-mappings in the context of complex valued metric spaces. The derived results generalize and extend some well known results in the literature. Examples are also given to demonstrate the validity of our results.

Keywords: Complex valued metric spaces, coincident point, fixed point, weakly compatible mappings, $(E.A)$ property, (CLR) property.

1 Introduction and Preliminaries

The most well-known fixed point result is Banach contraction principle [5], which is widely used result in fixed point theory. This principle has been studied and generalized in different spaces such as, rectangular metric spaces, semi metric spaces, pseudo metric spaces, Quasi metric spaces, D-metric spaces and cone metric spaces etc. and various fixed point theorems were developed.

Recently, Azam *et al.*[4] introduced the notion of complex valued metric space and established fixed point theorems for mappings satisfying a rational type inequality. Verma and Pathak [16] introduced the concept of $(E.A)$ and (CLR) properties in complex valued metric space to prove some common fixed point theorems for two pairs of weakly compatible mappings. Mohanta and Maitra [10] defined the concept of φ-mapping and proved a common fixed point theorem for a φ-pair in complex valued metric spaces. Subsequently, a number of fixed point theorems have been established and some applications have been discussed (see, for instance, [2], [3], [9], [11], [12], [14], [15])

In this article, using the concept $(E.A)$ and (CLR) properties we establish common fixed point theorems for weakly compatible mappings in the frame work of complex valued metric space.

We recall some basic definitions and results that we need in our discussion which can be found in [4]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

Consequently, one can say that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

1. $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;
2. $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;
3. $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$;
4. $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (1),(2) and (4) is satisfied and we will write $z_1 \prec z_2$ if only (3) is satisfied.

Note that

i. $a, b \in R$ and $a \leq b \Rightarrow az \precsim bz$, for all $z \in \mathbb{C}$;
ii. $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$;
iii. $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Azam *et al.*[4] defined the complex-valued metric space (X, d) in the following way:

Definition 1. Let X be a nonempty set and $d : X \times X \to \mathbb{C}$ be the mapping satisfying the following axioms:

1. $0 \precsim d(x,y)$, for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$, for all $x,y \in X$;
3. $d(x,y) \precsim d(x,z) + d(z,y)$, for all $x,y,z \in X$.

Then the pair (X,d) is called a complex valued metric space.

Example 1.1. [7] Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x,y) = e^{im}|x-y|,$$

where $x,y \in X$ and $0 \leq m \leq \frac{\pi}{2}$. Then (X,d) is a complex valued metric space.

Definition 2. [4] Let $\{x_n\}$ be a sequence in complex valued metric (X,d) and $x \in X$. Then x is called the limit of $\{x_n\}$ if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in N$ such that $d(x_n,x) \prec c$ for all $n > n_0$ and we write $\lim\limits_{n\to\infty} x_n = x$.

Lemma 1. *[4] Any sequence $\{x_n\}$ in complex valued metric space (X,d) converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Definition 3. [13] Let S and T be self-maps of a non empty set X. Then

1. $x \in X$ is a fixed point of T if $Tx = x$.
2. $x \in X$ is a coincidence point of S and T if $Sx = Tx$.
3. $x \in X$ is a common fixed point of S and T if $Sx = Tx = x$.

Jungck [8] initiated the concept of weakly compatible maps in ordinary metric spaces while Bhatt *et al.*[6] defined this notion in the complex valued metric space in the following way:

Definition 4. Let X be a complex valued metric space. Then a pair of self-mappings $S,T : X \to X$ are weakly compatible if they commute at their coincidence points, that is if there exists a point $x \in X$ such that $STx = TSx$ whenever $Sx = Tx$.

Aamri and Moutawakil [1] introduced property $(E.A)$ in ordinary metric spaces which generalized the notion of non-compatible mappings while Verma and Pathak [16] initiated this concept in complex valued metric space in the following way:

Definition 5. Let $T,S : X \to X$ be two self-maps on a complex valued metric space (X,d). Then the pair (T,S) is said to satisfy property $(E.A)$, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x \text{ for some } x \in X.$$

The notion of (CLR) property in ordinary metric spaces has been introduced by Sintunavarat and Kumam [13], in a similarly mode Verma and Pathak [16] defined this notion in a complex valued metric space in the following way:

Definition 6. Let $T,S : X \to X$ be two self-maps on a complex-valued metric space (X,d). Then T and S are said to satisfy the common limit in the range of S property, denoted by (CLR_S) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Sx \text{ for some } x \in X.$$

Mohanta and Maitra [10] introduced the concept of φ-mappings in complex valued matric space as:

Definition 7. Let $P = \{z \in \mathbb{C} : Re(z) \geq 0 \text{ and } Im(z) \geq 0\}$. A nondecreasing mapping $\varphi : P \to P$ is called a φ-mapping if

a. $\varphi(0) = 0$ and $0 \precnsim \varphi(z) \precnsim z$ for $z \in P \setminus \{0\}$,

b. $\varphi(z) \prec z$ for every $z \succ 0$,
c. $\lim_{n\to\infty} \varphi^n(z) = 0$ for every $z \in P \setminus \{0\}$.

Definition 8. [10] Let (X, d) be a complex valued metric space. The mappings $T, S : X \to X$ are called φ-pair if there exists a φ-mapping satisfying

$$d(Tx, Ty) \precsim \varphi(d(Sx, Sy)), \quad \text{for all } x, y \in X$$

Theorem 1.1. *[10] Let (X, d) be a complex valued metric space. Suppose that the mappings $T, S : X \to X$ satisfy*

$$d(Tx, Ty) \precsim \varphi(d(Sx, Sy)), \quad \textit{for all } x, y \in X,$$

where φ is φ-mapping. If $T(X) \subseteq S(X)$, T and S are weakly compatible and $T(X)$ or $S(X)$ is complete, then T and S have a unique common fixed point in X.

2 Main Results

Throughout this section $\mathbb{C}_+ = \{z \in \mathbb{C} : z \succsim 0\}$ and Ψ be the class of all lower semi-continuous functions $\psi : \mathbb{C}_+ \to \mathbb{C}_+$ such that $\psi(0) = 0$ and $\psi(z) \prec z$ for every $z \succ 0$.

Theorem 2.1. *Let (X, d) be a complex valued metric space and $K, L : X \to X$ be two self-mappings satisfying the following conditions:*

C_1. the pair (K, L) satisfies (CLR_L) property;
C_2. for each $x, y \in X$,

$$d(Kx, Ky) \precsim \psi(d(Lx, Ly)),$$

where $\psi \in \Psi$. Then K and L have a coincident point in X. Moreover, if the pair (K, L) is weakly compatible, then K and L have a unique common fixed point in X.

Proof. Assume that the pair (K, L) satisfies (CLR_L) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to+\infty} Kx_n = \lim_{n\to+\infty} Lx_n = Lu \ \text{ for some } \ u \in X. \tag{1}$$

Now, we show that $Ku = Lu$, for this putting $x = x_n$ and $y = z$ in condition (C_2) of Theorem 2.1, we have

$$d(Kx_n, Ku) \precsim \psi(d(Lx_n, Lu)),$$

taking lower limit as $n \to +\infty$, it follows that

$$\begin{aligned}\lim_{n\to+\infty} d(Kx_n, Ku) &\precsim \lim_{n\to+\infty} \psi(d(Lx_n, Lu)) \\ &= \psi\Big(\lim_{n\to+\infty} d(Lx_n, Lu)\Big),\end{aligned}$$

using equation (1), one can write

$$\begin{aligned}d(Lu, Ku) &\precsim \psi(d(Lu, Lu)) \\ &= \psi(0) = 0.\end{aligned}$$

Thus

$$Ku = Lu = z(say). \tag{2}$$

Since the pair (K, L) is weakly compatible, so that $Ku = Lu$ implies that $LKu = KLu$ and using equation (2), we get

$$Lz = Kz. \tag{3}$$

That is z is a coincident point of K and L in X.

Next, we show that z is a common fixed point of K and L in X. For this, we show that $Kz = z$. If $Kz \neq z$, then on using condition (C_2) of Theorem 2.1 with $x = u$ and $y = z$, we get

$$d(Ku, Kz) \precsim \psi(d(Lu, Lz)),$$

using equations (2) and (3), we have

$$d(z, Kz) \precsim \psi(d(z, Kz)) \prec d(z, Kz),$$

which is not possible, thus $Kz = z$ and hence from equation (3), we get

$$Kz = Lz = z. \tag{4}$$

That is z is a common fixed point of K and L in X.

Uniqueness: Assume that $z^* \neq z$ be another fixed point of K and L, i.e. $Kz^* = Lz^* = z^*$. Then on using condition (C_2) of Theorem 2.2, we have

$$\begin{aligned} d(z, z^*) &= d(Kz, Kz^*) \\ &\precsim \psi(d(Lz, Lz^*)) \\ &= \psi(d(z, z^*)) \\ &\prec d(z, z^*), \end{aligned}$$

which is a contradiction, thus $z = z^*$. Hence z is a unique common fixed point of K and L in X. □

If we replace (CLR) property by $(E.A)$ property in Theorem 2.1 we get the following result.

Theorem 2.2. *Let (X, d) be a complex valued metric space and $K, L : X \to X$ be two self-mappings satisfying the following conditions:*

C_1. *the pair (K, L) satisfies $(E.A)$ property;*
C_2. *for each $x, y \in X$,*

$$d(Kx, Ky) \precsim \psi(d(Lx, Ly)),$$

where $\psi \in \Psi$. If $L(X)$ is closed subspace of X, then K and L have a coincident point in X. Moreover, if the pair (K, L) is weakly compatible, then K and L have a unique common fixed point in X.

Proof. The proof easily follows from Theorem 2.1, so we omit it. □

Theorem 2.3. *Let (X, d) be a complex valued metric space and $K, L, M, N : X \to X$ be four self-mappings satisfying the following conditions:*

(C_1) *either the pair (K, M) satisfies (CLR_K) property or (L, N) satisfies (CLR_L) property;*
(C_2) *for each $x, y \in X$,*

$$d(Kx, Ly) \precsim \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)),$$

where $\psi \in \Psi$. If $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$, then each pair (K, M) and (L, N) have coincident point in X. Moreover, if the pairs (K, M) and (L, N) are weakly compatible, then K, L, M and N have a unique common fixed point in X.

Proof. Suppose that the pair (K, M) satisfies (CLR_K) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to+\infty} Kx_n = \lim_{n\to+\infty} Mx_n = Kx \quad \text{for some} \quad x \in X. \tag{5}$$

Since $K(X) \subseteq N(X)$, so there exists $u \in X$ such that $Kx = Nu$ and thus from (5), we get

$$\lim_{n\to+\infty} Kx_n = \lim_{n\to+\infty} Mx_n = Kx = Nu. \tag{6}$$

We claim that $Lu = Nu$. If not, then on putting $x = x_n$ and $y = u$ in condition (C_2) of Theorem 2.3, we get

$$d(Kx_n, Lu) \precsim \frac{1}{2}\psi(d(Kx_n, Nu) + d(Lu, Mx_n)),$$

taking lower limit as $n \to +\infty$, we have

$$\lim_{n\to+\infty} d(Kx_n, Lu) \precsim \frac{1}{2}\psi\left(\lim_{n\to+\infty} d(Kx_n, Nu) + \lim_{n\to+\infty} d(Lu, Mx_n)\right),$$

using equation (6), one can write

$$\begin{aligned} d(Nu, Lu) &\precsim \frac{1}{2}\psi(d(Lu, Nu)) \\ &\prec \frac{1}{2}d(Lu, Nu), \end{aligned}$$

which is not possible, thus $Lu = Nu$. But $L(X) \subseteq M(X)$, so there exists $v \in X$ such that $Lu = Mv$. Therefore

$$Lu = Nu = Mv = Kx. \tag{7}$$

Next, we show that $Kv = Mv$. Let $Kv \neq Mv$, then on using condition (C_2) of Theorem 2.3 with $x = v$ and $y = u$,

$$d(Kv, Lu) \precsim \frac{1}{2}\psi(d(Kv, Nu) + d(Lu, Mv)),$$

from equation (7) it follows that

$$\begin{aligned} d(Kv, Mv) &\precsim \frac{1}{2}\psi(d(Kv, Mv)) \\ &\prec \frac{1}{2}d(Kv, Mv), \end{aligned}$$

which is contradiction, thus $Kv = Mv$. From equation (7), we get

$$Kv = Lu = Nu = Mv = z(say). \tag{8}$$

Now, using the weak compatibility of the pairs (K, M), (L, N) and equation (8), we have

$$Kv = Mv \quad \Rightarrow KMv = MKv \quad \Rightarrow \quad Kz = Mz. \tag{9}$$

and

$$Nu = Lu \quad \Rightarrow LNu = NLu \quad \Rightarrow \quad Lz = Nz. \tag{10}$$

That is z is coincident point of each pair (K, M) and (L, N) in X.

In the next steps we prove that z is a common fixed point of K, L, M and N in X. For this, we have to show that $Kz = z$. If $Kz \neq z$, then on putting $x = z$ and $y = u$ in condition (C_2) of Theorem 2.3, one can get

$$d(Kz, Lu) \precsim \frac{1}{2}\psi(d(Kz, Nu) + d(Lu, Mz)),$$

using equations (8) and (9), we have

$$\begin{aligned} d(Kz, z) &\precsim \frac{1}{2}\psi(d(Kz, z) + d(z, Kz)) \\ &= \frac{1}{2}\psi(2d(Kz, z)) \\ &\prec d(Kz, z), \end{aligned}$$

which is not possible. Thus $Kz = z$ and hence from equation (9), we can write

$$Kz = Mz = z. \tag{11}$$

Similarly, putting $x = v$, $y = z$ in condition (C_2) of Theorem 2.3 and using equations (8) and (10), we get

$$Lz = Nz = z. \tag{12}$$

By combining equations (11) and (12), one can write

$$Kz = Lz = Nz = Mz = z. \tag{13}$$

That is z is a common fixed point of K, L, M and N in X.

Similarly, if we assume that the pair (L, N) satisfies (CLR_L) property. Then again we can show that z is a common fixed point of K, L, M and N in X.

Uniqueness: For the uniqueness of a common fixed point, we suppose that $z^* \neq z$ be another fixed point of K, L, M and N, i.e. $Kz^* = Lz^* = Mz^* = Nz^* = z^*$. Then on using condition (C_2) of Theorem 2.3, we get

$$\begin{aligned} d(z, z^*) &= d(Kz, Lz^*) \\ &\precsim \frac{1}{2}\psi(d(Kz, Nz^*) + d(Lz^*, Mz)) \\ &= \frac{1}{2}\psi(2d(z^*, z)) \\ &\prec d(z^*, z), \end{aligned}$$

which is contradiction, thus $z = z^*$. Hence z is a unique common fixed point of K, L, M and N in X. □

From above theorem one can easily derive the following corollaries.

Corollary 2.1. *Let (X, d) be a complex valued metric space and $L, M, N : X \to X$ be three self-mappings satisfying the following conditions:*

C_1. *either the pair (L, M) satisfies (CLR_L) property or (L, N) satisfies (CLR_L) property;*
C_2. *for each $x, y \in X$,*

$$d(Lx, Ly) \precsim \frac{1}{2}\psi(d(Lx, Ny) + d(Ly, Mx)),$$

where $\psi \in \Psi$. If $L(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$, then each pair (L, M) and (L, N) have coincident point in X. Moreover, if the pairs (L, M) and (L, N) are weakly compatible, then L, M and N have a unique common fixed point in X.

Corollary 2.2. *Let (X, d) be a complex valued metric space and $K, M : X \to X$ be two self-mappings satisfying the following conditions:*

C_1. *the pair (K, M) satisfies (CLR_K) property;*
C_2. *for each $x, y \in X$,*

$$d(Kx, Ky) \precsim \frac{1}{2}\psi(d(Kx, My) + d(Ky, Mx)),$$

where $\psi \in \Psi$. Then K and M have a coincident point in X. Moreover, if the pair (K, M) is weakly compatible, then K and M have a unique common fixed point in X.

If we replace (CLR) property by $(E.A)$ property in Theorem 2.3, we get the following result.

Theorem 2.4. *Let (X, d) be a complex valued metric space and $K, L, M, N : X \to X$ be four self-mappings satisfying the following conditions:*

C_1. *one of the pairs (K, M) and (L, N) satisfies $(E.A)$ property;*
C_2. *for each $x, y \in X$,*

$$d(Kx, Ly) \precsim \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)),$$

where $\psi \in \Psi$. If $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$ such that one of $M(X)$ and $N(X)$ is closed subspace of X, then each pair (K, M) and (L, N) have a coincidence point in X. Moreover, if the pairs (K, M) and (L, N) are weakly compatible, then K, L, M and N have a unique common fixed point in X.

Proof. The proof easily follows from Theorem 2.3, so we omit it. □

Similarly to Theorem 2.3 we can derive the corollary from Theorem 2.4.

3 Examples

In this section, we give two examples to illustrate our obtained results.

Example 3.1. Let $X = (-3, 1)$ be a complex valued metric space with metric $d : X \times X \to \mathbb{C}$ defined by $d(x, y) = e^{im}|x - y|$, where $x, y \in X$ and $0 \leq m \leq \frac{\pi}{2}$. Define self-maps K, L, M and N on X by:

$$Kx = \begin{cases} \frac{x+1}{3} & \text{if } x \in (-1, 0.5) \\ 0.5 & \text{if } x \in (-3, -1] \cup [0.5, 1) \end{cases};$$

$$Lx = \begin{cases} \frac{x-1}{2} & \text{if } x \in (-1, 0.5) \\ 0.5 & \text{if } x \in (-3, -1] \cup [0.5, 1) \end{cases};$$

$$Mx = \begin{cases} 10 & \text{if } x \in (-1, 0.5) \\ 0.5 & \text{if } x \in (-3, -1] \cup \{0.5\} \\ 1 - x & \text{if } x \in (0.5, 1) \end{cases};$$

and

$$Nx = \begin{cases} 0 & \text{if } x \in (-1, 0.5) \\ 0.5 & \text{if } x \in (-3, -1] \cup \{0.5\} \\ 2(1 - x) & \text{if } x \in (0.5, 1). \end{cases}$$

Then

$$K(X) = (0, 0.5], \quad L(X) = (-1, -0.25) \cup \{0.5\}, \quad M(X) = (0, 0.5] \cup \{10\}, \quad N(X) = [0, 1).$$

Also, define $\psi : \mathbb{C}_+ \to \mathbb{C}_+$ by $\psi(z) = \frac{z}{3}$.

First we check condition (C_1) of Theorem 2.3, for this let $\{x_n\} = \{\frac{n+1}{2n}\}_{n \geq 1}$ be a sequence in X. Then

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} K\left(\frac{n+1}{2n}\right) = \lim_{n\to\infty} 0.5 = 0.5,$$

and

$$\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} M\left(\frac{n+1}{2n}\right) = \lim_{n\to\infty} \left(1 - \frac{n+1}{2n}\right) = 0.5,$$

that is there exists a sequence $\{x_n\}$ in X such that

$$\begin{aligned} &\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Mx_n = 0.5 = Kx \text{ for all } \ x \in [0.5, 1). \\ \implies &\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Mx_n = 0.5 = Kx \text{ for some } \ x \in X. \end{aligned}$$

Hence (K, M) satisfies (CLR_K) property.

Next, to check condition (C_2) of Theorem 2.3, we discuss the following cases:
Case 1. let $x, y \in (-1, 0.5)$, then

$$Kx = \frac{x+1}{3}, \quad Ly = \frac{y-1}{2}, \quad Mx = 10, \quad \text{and} \quad Ny = 0.$$

Now,

$$d(Kx, Ly) = e^{im}\left|\frac{x+1}{3} - \frac{y-1}{2}\right| = e^{im}\left|\frac{2x - 3y + 5}{6}\right| \prec \frac{3}{2}e^{im},$$

and

$$\begin{aligned} \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)) =& \frac{1}{2}\psi\left(e^{im}\left|\frac{x+1}{3} - 0\right| + e^{im}\left|\frac{y-1}{2} - 10\right|\right) \\ =& \frac{1}{2}\psi\left(e^{im}\left\{\left|\frac{x+1}{3}\right| + \left|\frac{y-21}{2}\right|\right\}\right) \\ \succ& \frac{41}{24}e^{im}, \end{aligned}$$

so that

$$d(Kx, Ly) \prec \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)), \quad \text{for all} \quad x, y \in (-1, 0.5).$$

Case 2. Let $x, y \in (-3, -1] \cup \{0.5\}$, then $Kx = Mx = Ly = Ny = 0.5$ and hence

$$d(Kx, Ly) = 0 = \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)).$$

Case 3. Let $x, y \in (0.5, 1)$, then $Kx = Ly = 0.5, Mx = 1 - x$ and $Ny = 2(1 - y)$. Now

$$d(Kx, Ly) = 0,$$

and

$$\begin{aligned}\frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)) =& \frac{1}{2}\psi(e^{im}|Kx - Ny| + e^{im}|Ly - Mx|) \\ =& \frac{1}{2}\psi\bigg(e^{im}| - 1.5 + 2y| + |x - 0.5|\bigg),\end{aligned}$$

so that

$$d(Kx, Ly) \prec \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)), \quad \text{for all} \quad x, y \in (0.5, 1)$$

Therefore from the above three cases it follows that

$$d(Kx, Ly) \precsim \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)), \text{ for all } \ x, y \in X.$$

Also $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$, the pairs (K, M) and (L, N) are weakly compatible. Hence by Theorem (2.3), 0.5 is a unique common fixed point of K, L, M and N.

Example 3.2. Let $X = (0, 3] \cup \{4\}$ be a complex valued metric space with metric $d : X \times X \to \mathbb{C}$ defined by $d(x, y) = e^{im}|x - y|$, where $x, y \in X$ and $0 \le m \le \frac{\pi}{2}$. Define self-maps K, L, M and N on X by:

$$Kx = \begin{cases} \frac{4}{3} \text{ if } x \in (0, 1) \\ 1 \text{ if } x \in [1, 3] \end{cases};$$

$$Lx = \begin{cases} \frac{1}{2} \text{ if } x \in (0, 1) \\ 1 \text{ if } x \in [1, 3] \end{cases};$$

$$Mx = \begin{cases} 4 & \text{if } x \in (0, 1) \\ 1 & \text{if } x = 1 \\ \frac{x-1}{2} & \text{if } x \in (1, 3] \end{cases};$$

and

$$Nx = \begin{cases} \frac{2}{3} & \text{if } x \in (0, 1) \\ 1 & \text{if } x = 1 \\ \frac{x+1}{3} & \text{if } x \in (1, 3]. \end{cases}$$

Also, define $\psi : \mathbb{C}_+ \to \mathbb{C}_+$ by $\psi(z) = \frac{z}{2}$. Then

$$K(X) = \left\{1, \frac{4}{3}\right\}, \quad L(X) = \left\{1, \frac{1}{2}\right\}, \quad M(X) = (0, 1] \cup \{4\}, \quad N(X) = \left[\frac{2}{3}, \frac{4}{3}\right].$$

Now, Let $\{x_n\} = \{3 - \frac{1}{n}\}_{n \ge 1}$ be a sequence in X. Then

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} K\left(3 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 = 1,$$

and

$$\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} M\left(3 - \frac{1}{n}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{2n}\right) = 1.$$

Thus there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Mx_n = 1 \in X.$$

Hence (K, M) satisfies $(E.A)$ property.

To check condition (C_2) of Theorem 2.4, we distinguish the following cases:
Case 1. let $x, y \in (0, 1)$, then $Kx = \frac{4}{3}$, $Ny = \frac{2}{3}$, $Ly = \frac{1}{2}$ and $Mx = 4$.
Now

$$d(Kx, Ly) = \frac{5}{6}e^{im},$$

and

$$\begin{aligned}\frac{1}{2}\psi(d(Kx,Ny)+d(Ly,Mx)) =&\frac{1}{2}\psi\left(e^{im}\left|\frac{4}{3}-\frac{2}{3}\right|+e^{im}\left|\frac{1}{2}-4\right|\right)\\ =&\frac{1}{2}\psi\left(\frac{25}{6}e^{im}\right)=\frac{1}{2}\left(\frac{25}{12}e^{im}\right)=\frac{25}{24}e^{im}\\ \succ&\frac{5}{6}e^{im}\\ =&d(Kx,Ly).\end{aligned}$$

Thus

$$d(Kx, Ly) \prec \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)), \quad \text{for all} \quad x, y \in (0, 1),$$

Case 2. let $x = y = 1$, then $Kx = Mx = Ly = Ny = 1$ and hence

$$d(Kx, Ly) = 0 = \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)),$$

Case 3. let $x, y \in (1, 3]$, then $Kx = Ly = 1, Mx = \frac{x-1}{2}$ and $Ny = \frac{y+1}{3}$.
Now

$$d(Kx, Ly) = 0,$$

and

$$\begin{aligned}\frac{1}{2}\psi(d(Kx,Ny)+d(Ly,Mx)) =&\frac{1}{2}\psi\left(e^{im}\left\{\left|1-\frac{y+1}{3}\right|+\left|1-\frac{x-1}{2}\right|\right\}\right)\\ =&\frac{1}{2}\left(2e^{im}\left\{\left|\frac{2-y}{3}\right|+\left|\frac{3-x}{2}\right|\right\}\right),\end{aligned}$$

so that

$$d(Kx, Ly) \precsim \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)), \text{ for all } \ x, y \in (1, 3].$$

Therefore from the above three cases it follows that

$$d(Kx, Ly) \precsim \frac{1}{2}\psi(d(Kx, Ny) + d(Ly, Mx)), \text{ for all } \ x, y \in X.$$

Also $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$ such that $N(X)$ is closed subspace of X and the pairs (K, M) and (L, N) are weakly compatible. Hence by Theorem 2.4, we can say that 1 is a unique common fixed point of K, L, M and N.

4 Conclusions

The derived results extend and generalize some well known results of the existing literature in the frame work of compex valued metric spaces. In particular Theorems 2.2 and 2.1 generalize Theorem 1.1. We observe that $(E.A)$ property requires the condition of closedness of the subspace, while (CLR) property never requires any condition of closedness (or completeness) of the subspace. As we note that in Example 3.1 neither $M(X)$ nor $N(X)$ is closed subspaces of X, so Theorem 2.4 is not applicable.

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