Gegenbauer Transformations Nikolski-Besov Spaces Generalized by Gegenbauer Operator and Their Approximation Characteristics

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Abstract In this paper we consider some problems of the theory of approximation of functions on interval $[0, \infty)$ in the metric of $L_{p,\lambda}$ with weight $sh^{2\lambda} x$ using generalized Gegenbauer shifts. We prove analogues of direct Jackson theorems for the modulus of smoothness of arbitrary order defined in terms of generalized Gegenbauer shifts. We establish the equivalence of the modulus of smoothness and K-functional, defined in terms of the space of the Sobolev type corresponding to the Gegenbauer differential operator. We define function spaces of Nikol'skii-Besov type and describe them in terms of best approximations. As a tool for approximation, we use some functions classes of spectrum. In these classes, we prove analogues of Bernstein's inequality and others for the Gegenbauer differential operator. Our results are analogues of the results for generalized Bessel shifts obtained in the work [30].

Keywords: Approximation of functions, generalized Gegenbauer shift, Gegenbauer transformation, Nikol'skii-Besov type spaces, embedding theorems.

In the classical theory of approximation of functions on $\mathbb{R} = (-\infty, \infty)$ the shift operator $f(x) \mapsto f(x + y)$, $x, y \in \mathbb{R}$. plays a central role. This shift operator is used in the construction of the moduli of continuity and smoothness, which are the basic elements of the direct and inverse theorems of approximation theory. Various generalizations of shift operators enable to state natural analogues of problems in approximation theory. Groups and semigroups of operators on Banach spaces are generalizations of the shift operator. Different problems of approximation theory on Banach spaces with groups and semigroups of operators were considered in [1, 3, 5, 7, 44].

Generalized shift operator naturally follows from "addition theorem" for eigen functions of the differential operators (for example, Legendre, Gegenbauer, Jacobi, Laguerre, Hermite and other). These operators may not form a group or semigroup, but the generalized moduli of smoothness defined in terms of them can be better adapted to the study of relations between the smoothness properties of functions and the best approximations of these functions in weighted function spaces. Some results on the best approximation of functions using generalized shift operators can be found in [1, 4, 22-26, 29-39, 45]. Note that most of the papers on this topic deal with the approximation of functions by polynomials on a finite segment. For the half-line, most popular examples are the generalized Bessel and Dunkl shifts (see for example [4, 23, 30-32]). Fourier-Bessel and Fourier-Dunkl harmonic analysis, which deals with Bessel and Dunkl integral transformations and their applications, are closely connected with the generalized Bessel and Dunkl shift. Moreover, generalized Bessel shift is widely used in the potential theory and theory of maximal functions (see, example [5, 11-13]). The file of constructions of the theory of generalized shift operators generalize in the theory of transformation of operators (see for example [7]). We reduce only these works which have at least some relation to the paper. In this paper the generalized Gegenbauer shift are considered and by using this some questions of approximation theory of functions in $[0, \infty)$ in the metric $L_{p,\lambda}$ with weight $sh^{2\lambda} x$ are studied.

1   The Properties of Generalized Gegenbauer Shift and Statement of Main Results

For every p, $1 \leq p < \infty$, we denote by $L_{p,\lambda}$ the Banach space of measurable functions f on $\mathbb{R}_+$ (defined up to their values on a set of measure zero) such that the norm

$$\|f\|_{p,\lambda} \equiv \|f(ch)(\cdot)\|_{p,\lambda} = \left(\int_0^\infty |f(ch\,t)|^p \, sh^{2\lambda} \, t\mathrm{d}t \right)^{\frac{1}{p}}$$

is finite.

For $p = \infty$ we denote by $L_{\infty,\lambda}$ the set of all functions f that are uniformly continuous and bounded on $\mathbb{R}_+$. The norm in $L_{\infty,\lambda}$ is defined by the formula

$$\|f\|_{\infty,\lambda} \equiv \|f(ch)(\cdot)\|_{\infty,\lambda} = \sup_{t \in \mathbb{R}_+} |f(ch\,t)|.$$

Let

$$D_\lambda = \left(x^2 - 1\right)^{\frac{1}{2}-\lambda} \frac{\mathrm{d}}{\mathrm{d}x} \left(x^2 - 1\right)^{\lambda+\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}x} \tag{1.1}$$

be Gegenbauer differential operators.

The functions (see [9], for.(2.3) and (2.8), and also [11], p.1045, for. 8.936(1))

$$
\begin{aligned}
P_\alpha^\lambda (ch\,x) &= \frac{\Gamma(\alpha + 2\lambda)\cos\lambda\pi}{\Gamma(\lambda)\,\Gamma(\alpha + \lambda + 1)} (2ch\,x)^{-\alpha-2\lambda} \\
&\times F\left(\frac{\alpha}{2} + \lambda, \; \frac{\alpha}{2} + \lambda + \frac{1}{2}; \; \alpha + \lambda + 1; \; \frac{1}{ch^2\,x} \right), \; x \geq 0,
\end{aligned}
\tag{1.2}
$$

$$
C_\alpha^\lambda (ch\,x) = \sqrt{\pi}\,\frac{\Gamma(\alpha + 2\lambda)}{\Gamma(\lambda)\,\Gamma(\alpha + 1)} (2sh\,x)^{\frac{1}{2}-\lambda} P_{\alpha+\lambda-1}^{\frac{1}{2}-\lambda}(ch\,x)
$$

$$
= \frac{\Gamma(\alpha + \lambda)}{\Gamma(\lambda)\,\Gamma(\alpha + 1)} (2ch\,x)^\alpha F\left(-\frac{\alpha}{2}, \; -\frac{\alpha}{2} + \frac{1}{2}; \; 1 - \alpha - \lambda; \; \frac{1}{ch^2\,x} \right), \tag{1.3}
$$

where $P_{\alpha+\lambda-1}^{\frac{1}{2}-\lambda}(ch\,x)$ is Legendre function, and $F(\alpha, \beta; \gamma; x)$ is Gauss's hypergeometric function, are the linearly independent solutions of equation

$$sh^2x\,y''(ch\,x) + (2\lambda + 1)\,ch\,x\,y'(ch\,x) - \alpha(\alpha + 2\lambda)\,y(ch\,x) = 0. \tag{1.4}$$

For the functions $P_\alpha^\lambda(ch\,x)$ and $C_\alpha^\lambda(ch\,x)$ the following formulas (see [9], p. 1939) are valid:

$$P_\alpha^\lambda (ch\,x\,ch\,t - sh\,x\,sh\,t\,\cos\varphi)$$

$$= \frac{\Gamma(2\lambda - 1)}{\Gamma^2(\lambda)} \sum_{n=0}^\infty (-1)^n \frac{4^n \Gamma(\alpha - n + 1)\,\Gamma^2(\lambda + n)\,(2n + 2\lambda - 1)}{\Gamma(\alpha + 2\lambda + n)} \tag{1.5}$$

$$\times sh^n\,x\,sh^n\,t\,P_{\alpha-n}^{\lambda+n}(ch\,x)\,C_{\alpha-n}^{\lambda+n}(ch\,t)\,C_n^{\lambda-\frac{1}{2}}(\cos\varphi),$$

$$C_\alpha^\lambda (ch\,x\,ch\,t - sh\,x\,sh\,t\,\cos\varphi)$$

$$= \frac{\Gamma(2\lambda - 1)}{\Gamma^2(\lambda)} \sum_{n=0}^\infty (-1)^n \frac{4^n \Gamma(\alpha - n + 1)\,\Gamma^2(\alpha + n)\,(2n + 2\lambda - 1)}{\Gamma(a + 2\lambda + n)} \tag{1.6}$$

$$\times sh^n\,x\,sh^n\,t\,C_{\alpha-n}^{\lambda+n}(ch\,x)\,C_{\alpha-n}^{\lambda+n}(ch\,t)\,C_n^{\lambda-\frac{1}{2}}(\cos\varphi),$$

where $\alpha - n \neq -1, -2, \ldots$.

Taking into account the following equation (see [11], p.844)

$$\int_0^\pi C_n^{\lambda-\frac{1}{2}} (\cos \varphi) (\sin \varphi)^{2\lambda-1} \, \mathrm{d}\varphi = \begin{cases} 0, & n \geq 1, \\ \dfrac{\Gamma(\lambda)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right)}, & n = 0 \end{cases}$$

from (1.5) and (1.6) we obtain

$$A_{ch\ t}^\lambda P_\alpha^\lambda (ch\ x) = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\,\Gamma\left(\frac{1}{2}\right)} \int_0^\pi P_\alpha^\lambda (ch\ x\ ch\ t - sh\ x\ sh\ t \cos \varphi) (\sin \varphi)^{2\lambda-1} \, \mathrm{d}\varphi \tag{1.7}$$

$$= \frac{\Gamma(2\lambda)\,\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda)} P_\alpha^\lambda(ch\ t) C_\alpha^\lambda (ch\ x) = P_\alpha^\lambda(ch\ t) Q_\alpha^\lambda (ch\ x),$$

$$A_{ch\ t}^\lambda Q_\alpha^\lambda (ch\ x) = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\,\Gamma\left(\frac{1}{2}\right)} \int_0^\pi Q_\alpha^\lambda (ch\ x\ ch\ t - sh\ x\ sh\ t \cos \varphi) (\sin \varphi)^{2\lambda-1} \, \mathrm{d}\varphi \tag{1.8}$$

$$= Q_\alpha^\lambda(ch\ t) Q_\alpha^\lambda (ch\ x).$$

In (1.7) and (1.8)

$$Q_\alpha^\lambda (ch\ x) = \frac{\Gamma(2\lambda)\,\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda)} C_\alpha^\lambda (ch\ x),$$

where

$$A_{ch\ t}^\lambda f (ch\ x) \equiv A_{ch\ t} f (ch\ x)$$

$$= C(\lambda) \int_0^\pi f (ch\ x\ ch\ t - sh\ x\ sh\ t \cos \varphi) (\sin \varphi)^{2\lambda-1} \, \mathrm{d}\varphi, \tag{1.9}$$

and

$$C(\lambda) = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\,\Gamma\left(\frac{1}{2}\right)} = \left(\int_0^\pi (\sin \varphi)^{2\lambda-1} \, \mathrm{d}\varphi\right)^{-1}$$

are generalized Gegenbauer shift operator, follows from "addition theorem" (1.5) and (1.6) for Gegenbauer functions naturally.

Here and further after $c(a, b, ...) > 0$ will denote the constants, dependent only on written parameter, moreover this perhaps differs in different of formulas.

The generalized Gegenbauer shift operator possesses some properties which are analogues to generalized Bessel shift operator in the work of Levitan [26]:

1) Linearity and homogeneity:

$$A_{ch\ t} \{a\, f(ch\ x) + b g(ch\ x)\} = a A_{ch\ t}\, f(ch\ x) + b A_{ch\ t} g(ch\ x),$$

which follows from the integral property.

2) Positivity. $A_{ch\ t} f(ch\ x) \geq 0$, if $f(ch\ x) \geq 0$, which is evident.

3) $A_1 f(ch\ x) = f(ch\ x)$, $A_{ch\ t} 1 = 1$, which follows from (1.9).

4) If $f(ch\ x) \equiv 0$ for $x \geq a$, then $A_{ch\ t} f(ch\ x) \equiv 0$ for $|x - t| \geq a$.

Indeed, $ch\ x\ ch\ t - sh\ x\ sh\ t \cos \varphi \geq ch\ (x - t) \geq |x - t|$, from this the property 4) follows.

5) $L_{p,\lambda}$-boundedness. For any $f \in L_{p,\lambda}$, $1 \leq p \leq \infty$, (see [14], Lemma 2)

$$\|A_{ch\ t} f\|_{p,\lambda} \leq \|f\|_{p,\lambda}, \quad t \in [0, \infty). \tag{1.10}$$

6) The operator $A_{ch\ t}$ is continuous on $L_{p,\lambda}$.

7) Symmetry of the operator $A_{ch\ t}$,

$$A_{ch\ t} f (ch\ x) = A_{ch\ x} f(ch\ t)$$

is evident.

8) Commutativity. For every continuous functions $f(ch\ x)$ and any really $y, t \geq 0$ the following equality is valid:

$$A_{ch\ y}A_{ch\ t}f(ch\ x) = A_{ch\ t}A_{ch\ y}f(ch\ x).$$

9) The operator $A_{ch\ t}$ is self-adjoint (see [14], Lemma 3). This property is proved in [13].

For any $f, g \in L_{1,\lambda}$ the following equality is valid:

$$\int_0^\infty A_{ch\ t}f(ch\ x)\, g(ch\ t)\, sh^{2\lambda}\, t\mathrm{d}t = \int_0^\infty f(ch\ t)\, A_{ch\ t}g(ch\ x)\, sh^{2\lambda}\, t\mathrm{d}t.$$

For every $f \in L_{p,\lambda}$ the differences of order k $(k = 1, 2, ...)$ are defined by the formula

$$\Delta_{ch\ t}^1 f(ch\ x) = \Delta_{ch\ t}f(ch\ x) := A_{ch\ t}f(ch\ x) - f(ch\ x),$$

$$\Delta_{ch\ t}^k f(ch\ x) := A_{ch\ t}\left(\Delta_{ch\ t}^{k-1}f(ch\ x)\right),\ k = 2, 3, ...$$

or

$$\Delta_{ch\ t}^k f(ch\ x) := \sum_{\nu=0}^k (-1)^{k-\nu} \binom{k}{\nu} A_{ch\ t}^\nu f(ch\ x). \tag{1.11}$$

For every natural number k the modulus $\omega_k(f, \delta)_{p,\lambda}$ of order k in $L_{p,\lambda}$ are defined by the formula

$$\omega_k(f, \delta)_{p,\lambda} := \sup_{0 < t \leq \delta} \left\|\Delta_{ch\ t}^k f\right\|_{p,\lambda}. \tag{1.12}$$

We say that the function $f \in L_{p,\lambda}$ has bounded spectrum of order ν, if $\hat{f}_p(\alpha) = 0$ for $\alpha > \nu$ (the definition $\hat{f}_p(\alpha)$ see lower the formula (2.1)).

We denote by $D_\lambda^k[0, \infty)$ the class of functions which we apply the operator D_λ to them $k-$times, i.e., $\exists D_\lambda^k f,\ k = 1, 2, ...$.

Let $W_{p,\lambda}^m,\ m = 1, 2, \ldots$, be the Sobolev space constructed for the differential operator D_λ, that is

$$W_{p,\lambda}^m := \left\{f \in L_{p,\lambda} : D_\lambda^k f \in L_{p,\lambda},\ k = 1, 2, .., m\right\}. \tag{1.13}$$

We denote by $\mathfrak{M}(\nu, p, \lambda),\ \nu > 0$ the set of functions $\Phi(ch\ x), x \in \mathbb{R}$, which satisfies the following conditions:

1) $\Phi(ch\ x)$ is the spectrum of order ν;

2) $\Phi(ch\ x)$ belongs to the class $W_{p,\lambda}^m$.

The best approximation of an $f \in L_{p,\lambda}$ by functions belonging to $\mathfrak{M}(\nu, p, \lambda)$ is defined as follows:

$$E_\nu(f) := \inf\left\{\|f - \Phi\|_{p,\lambda} : \Phi \in \mathfrak{M}(\nu, p, \lambda)\right\}. \tag{1.14}$$

The symbol $0 \leq a \lesssim b$ means that $a \leq cb$, where $c-$some constant, which can dependent on some parameters. The following theorem is an analogue of Jackson's direct theorem in classical approximation theory (see [41], ch. 5).

Theorem 1.1. Let $f \in W_{p,\lambda}^m$. Then for all $\nu > 0$ and $m \in \mathbb{Z}_+ = \{0, 1, 2, ...\}$, the inequality

$$E_\nu(f)_{p,\lambda} \lesssim \nu^{-2m}\omega_k\left(D_\lambda^m f, \frac{1}{v}\right)_{p,\lambda}$$

holds.

We define the K-functional Peetre construction for $L_{p,\lambda}$ and $W_{p,\lambda}^m$ by the formula:

$$K\left(f, t; L_{p,\lambda}; W_{p,\lambda}^m\right) := \inf\left\{\|f - g\|_{p,\lambda} + t\|D_\lambda^m g\|_{p,\lambda} : g \in W_{p,\lambda}^m\right\},$$

where $f \in L_{p,\lambda}$ and $t > 0$.

The symbol $\alpha(t) \simeq \beta(t)$ at $t \to 0$ means that there exists constants c_1 and c_2 such that

$$c_1\alpha(t) \leq \beta(t) \leq c_2\alpha(t).$$

The next theorem shows that the modulus of smoothness is equivalent to the K-functional.

Theorem 1.2. *For every $\delta > 0$ the following correlation*

$$K\left(f, \delta^{2m}; L_{p,\lambda}; W_{p,\lambda}^m\right) \simeq \omega_m\left(f, \delta\right)_{p,\lambda}, \quad t \to 0 \tag{1.15}$$

holds.

The proofs of inverse theorems in approximation theory are based on various analogues of Bernstein's inequality. In this paper we obtain such analogues for the Gegenbauer differential operator D_λ.

Theorem 1.3. *The following inequality is valid:*

$$\|D_\lambda f\|_{p,\lambda} \leq \nu^2 \|f\|_{p,\lambda} \tag{1.16}$$

for all $f \in \mathfrak{M}\left(\nu, p, \lambda\right)$.

Inequality (1.16) is an analogue of the classical Bernstein inequality for entire functions of exponential type (see [28], p.117).

Let $r > 0$ be a real number and let k, m be non-negative numbers such that $2k > r - 2m > 0$. We denote by $H_{p,\lambda}^r$ the set of all $f \in W_{p,\lambda}^m$ such that and

$$\omega_k\left(D_\lambda^m, \delta\right)_{p,\lambda} \leq A_f \delta^{r-2m}, \quad \delta > 0 \tag{1.17}$$

for some $A_f > 0$. For $f \in H_{p,\lambda}^r$ we define the seminorm as

$$h_{p,\lambda}^r\left(f\right) := \sup_{\delta > 0} \frac{\omega_k\left(D_\lambda^m, \delta\right)_{p,\lambda}}{\delta^{r-2m}}. \tag{1.18}$$

$H_{p,\lambda}^r$ is a Banach space with norm (see section 5)

$$\|f\|_{H_{p,\lambda}^r} := \|f\|_{p,\lambda} + h_{p,\lambda}^r\left(f\right). \tag{1.19}$$

In the next theorem we describe the space $H_{p,\lambda}^r$ in terms of approximations by functions belonging to $\mathfrak{M}\left(\nu, p, \lambda\right)$. In particular, this theorem implies that the $H_{p,\lambda}^r$ does not depend on k or m.

Theorem 1.4. *If $f \in H_{p,\lambda}^r$, then*

$$E_\nu\left(f\right)_{p,\lambda} \lesssim \nu^{-r} h_{p,\lambda}^r\left(f\right) \tag{1.20}$$

for $\nu \geq 1$.

Conversely, if $f \in L_{p,\lambda}$, then

$$E_\nu\left(f\right)_{p,\lambda} \leq \frac{A_f}{\nu^r} \tag{1.21}$$

for $\nu \geq 1$, where A_f is a constant depending on f.

Let $1 \leq q \leq \infty$, $r > 0$, and let k, m be non-negative integers such that $2k > r - 2m > 0$. As in [27], we say that a function f belongs to the Nikol'skii-Besov class $B_{p,q,\lambda}^r \equiv B_{p,q,\lambda}^r(\mathbb{R}_+)$ if $f \in W_{p,\lambda}^m$ and the seminorm

$$b_{p,q,\lambda}^r\left(f\right) := \begin{cases} \left(\int\limits_0^\infty \frac{\left(\omega_k\left(D_\lambda^m f, \delta\right)_{p,\lambda}\right)^q}{\delta^{(r-2m)q}} \frac{\mathrm{d}\delta}{\delta}\right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \sup\limits_{\delta > 0} \frac{\omega_k\left(D_\lambda^m f, \delta\right)_{p,\lambda}}{\delta^{r-2m}}, & q = \infty \end{cases}$$

is finite. Note that $B_{p,q,\lambda}^r$ is a Banach space with norm (see section 5)

$$\|f\|_{B_{p,q,\lambda}^r} := \|f\|_{p,\lambda} + b_{p,q,\lambda}^r. \tag{1.22}$$

Note that $B_{p,\infty,\lambda}^r = H_{p,\lambda}^r$. Let $\mathbb{Z}_+ = \{0, 1, 2, ...\}$.

Theorem 1.5. *Let $a > 1$ be an arbitrary number. The function $f \in L_{p,\lambda}$ belongs to $B_{p,q,\lambda}^r$ if and only if the seminorm*

$$\tilde{b}_{p,q,\lambda}^r := (f) \begin{cases} \left(\sum_{n=0}^{\infty} a^{nrq} \left(E_{a^n} (f)_{p,\lambda} \right)^q \right)^{\frac{1}{q}}, \ 1 \le q < \infty, \\ \sup_{n \in \mathbb{Z}_+} a^{nr} E_{a^n} (f)_{p,\lambda}, \ \ q = \infty \end{cases}$$

is finite. In this case the norm (1.21) is equivalent to the norm

$$\|f\|_{p,\lambda} + \tilde{b}_{p,q,\lambda}^r (f). \tag{1.23}$$

The main purpose of this paper is to prove Theorems 1.1 - 1.5. In section 2 we study various properties of Gegenbauer transformation in generalized functions classes, and also obtain some properties of the convolution of functions in these classes. In particular we prove that the Gegenbauer transformation and the convolution defined in the generalized function class are of generalized functions, i.e., they are linear and continuous functions.

The paper is organized as follows:

Some auxiliary results are stared is Section 3 althogh some of them have independent interest. For example, inequality (3.50) is the analogue of classical Nikol'skii-Stechkin inequality. From (3.51), assertion of Theorem 1.3, which is an analogue of classical Bernstein's inequality, follows. Proposition 3.17 and its corollary are analogues of the classical Boas inequality. In section 4 Theorem 1.1 and 1.2 are proved. In section 5 Theorem 1.4 and 1.5 are proved and also obtained various norms of the spaces $B_{p,q,\lambda}^r$.

2   The Gegenbauer Transformations

The Gegenbauer transformations of the functions $P_\alpha^\lambda (t)$ and $Q_\alpha^\lambda (t)$ are defined as the following integral transformations (see [18]):

$$F_P f (t) \mapsto \hat{f}_P(\alpha) = \int_1^\infty f (t) P_\alpha^\lambda (t) \left(t^2 - 1 \right)^{\lambda - \frac{1}{2}} dt, \tag{2.1}$$

$$F_Q f (t) \mapsto \hat{f}_Q(\alpha) = \int_1^\infty f (t) Q_\alpha^\lambda (t) \left(t^2 - 1 \right)^{\lambda - \frac{1}{2}} dt. \tag{2.2}$$

The inverse Gegenbauer transformations are defined by the following formulas:

$$F_P^{-1} \hat{f}_P(\alpha) \mapsto f (x) = C_\lambda \int_1^\infty \hat{f}_P(\alpha) Q_\alpha^\lambda (x) \left(\alpha^2 - 1 \right)^{\lambda - \frac{1}{2}} d\alpha, \tag{2.3}$$

$$F_Q^{-1} \hat{f}_Q(\alpha) \mapsto f (x) = C_\lambda \int_1^\infty \hat{f}_Q(\alpha) P_\alpha^\lambda (x) \left(\alpha^2 - 1 \right)^{\lambda - \frac{1}{2}} d\alpha, \tag{2.4}$$

where

$$C_\lambda = \frac{2^{\frac{3}{2} - \lambda} \Gamma \left(\frac{1}{2} \right) \Gamma (\lambda + 1) \Gamma \left(\frac{1}{2} - \lambda \right) \Gamma \left(\frac{3}{4} + \frac{\lambda}{2} \right) \left(\Gamma \left(\lambda + \frac{1}{2} \right) \Gamma \left(\frac{5}{4} - \frac{\lambda}{2} \right) \cos \pi \lambda \right)^{-1}}{{}_2F_1 \left(1, \frac{1}{2} - \lambda; \frac{5}{4} - \frac{\lambda}{2}; \frac{1}{2} \right) - {}_2F_1 \left(1, \frac{1}{2} - \lambda; \frac{5}{4} - \frac{\lambda}{2}; \frac{1}{4} - \frac{\lambda}{2} \right)}.$$

The $\hat{f}_P$ transformation (2.1) for any $f \in L_{p,\lambda}$ $(1 \le p \le \infty)$ exists that follows from the following inequality (see [18], Lemma 5)

$$|\hat{f}_P (\alpha)| \lesssim \|f\|_{p,\lambda} \alpha^{2\lambda - 1}. \tag{2.5}$$

From this it follows that for any $f \in L_{p,\lambda}$ $(1 \le p \le \infty)$ $\lim_{n \to \infty} \hat{f}_P (\alpha) = 0$. Particulary for $p = 1$ we obtain the analogue of Riemann-Lebesgue lemma. On the other hand (see further lemma 3.3), if f and $D_\lambda^k f$ belong to $L_{p,\lambda}$, then

$$\widehat{\left(D_\lambda^k f \right)}_P (\alpha) = (\alpha (\alpha + 2\lambda))^k \hat{f}_P (\alpha) \ \ k = 1, 2, \dots. \tag{2.6}$$

From (2.5) and (2.6) we have

$$|\hat{f}_P(\alpha)| \lesssim \|D_\lambda^k f\|_{p,\lambda} \alpha^{2\lambda-1} (\alpha(\alpha+2\lambda))^{-k}. \tag{2.7}$$

From this it follows that

$$\|\hat{f}_P\|_{p,\lambda}^p = \int_1^\infty |\hat{f}_P(\alpha)|^p (\alpha^2-1)^{\lambda-\frac{1}{2}} \, d\alpha$$

$$\lesssim \int_1^\infty \alpha^{2\lambda-1} (\alpha^2-1)^{\lambda-\frac{1}{2}} \alpha^{-2k} d\alpha \lesssim \int_1^\infty \alpha^{\lambda-k-1} (\alpha-1)^{\lambda-\frac{1}{2}} \, d\alpha$$

$$= \int_0^\infty \alpha^{\lambda-\frac{1}{2}} (\alpha+1)^{\lambda-k-1} \, d\alpha = \frac{\Gamma\left(\lambda+\frac{1}{2}\right) \Gamma\left(k-2\lambda+\frac{1}{2}\right)}{\Gamma(k-\lambda+1)}.$$

From this and from (2.5) we obtain that $\hat{f}_P \in L_{p,\lambda}$ for $1 \le p \le \infty$, as

$$\|\hat{f}_P\|_{\infty,\lambda} = \sup_{1 \le \alpha < \infty} |\hat{f}_P(\alpha)| \lesssim \|f\|_{p,\lambda}.$$

Consider the function $\hat{f}_Q$ in transformation (2.2). It is clear that for any function in $L_{p,\lambda}$ $\hat{f}_Q$ can not exist from the formula (1.3).

Let S be the space of test functions on $[1,\infty)$, that is, S is the set of all infinitely differentiable functions $\varphi(t)$, decaying as $t \to \infty$, to zero together with all their derivatives, more rapidly than any power of t^{-1}. In this way for any test function $\varphi(t)$ the following inequality (see [10], p.7) holds

$$|D^q \varphi(t)| \le C_{qr} (1+t^2)^{-r}$$

with any integer $q \ge 0$ and $r \ge 0$, where the constant C_{qr} does not depend on t.

It is clear that the operator D_λ can apply to test functions infinite times, moreover the estimations

$$|D_\lambda^m \varphi(t)| \lesssim (1+t^2)^{-r} \tag{2.8}$$

hold for arbitrary integer $m \ge 0$ and $r \ge 0$.

From (1.4) it follows that

$$D_\lambda^m Q_\alpha^\lambda(x) = (\alpha(\alpha+2\lambda))^m Q_\alpha^\lambda(x), \quad \alpha \in [1,\infty), \ m = 1, 2\dots.$$

From this we have

$$Q_\alpha^\lambda(x) = (\alpha(\alpha+2\lambda))^{-m} D_\lambda^m Q_\alpha^\lambda(x).$$

In consideration of this equality in (2.2), we obtain

$$\hat{f}_Q(\alpha) = (\alpha(\alpha+2\lambda))^{-m} \int_1^\infty f(t) D_\lambda^m Q_\alpha^\lambda(t) (t^2-1)^{\lambda-\frac{1}{2}} \, dt$$

$$= (\alpha(\alpha+2\lambda))^{-m} \int_1^\infty Q_\alpha^\lambda(t) D_\lambda^m f(t) (t^2-1)^{\lambda-\frac{1}{2}} \, dt, \tag{2.9}$$

where we use the self-adjointness of the operator D_λ (see [14], Lemma 4).

Applying inequality (2.8) at $r > \alpha+1$ in (2.9) we have

$$|\hat{f}_Q(\alpha)| \lesssim (\alpha(\alpha+2\lambda))^{-m} \int_1^\infty \frac{Q_\alpha^\lambda(t) (t^2-1)^{\lambda-\frac{1}{2}}}{(1+t^2)^{\alpha+1}} \, dt. \tag{2.10}$$

The Gauss's hypergeometric function $_2F_1\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}+\frac{1}{2}; 1-\alpha-\lambda; x^{-2}\right)$ in the formula (1.3) exists by appointed meaning of the parameters for all $x \in [1,\infty)$ (see [11], p.1054). Therefore from (1.3) we have the estimate

$$Q_\alpha^\lambda(x) = \frac{\Gamma(2\lambda) \Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda)} C_\alpha^\lambda(x) \lesssim \alpha^{-\lambda} (2x)^\alpha.$$

Taking into account this inequality in (2.10) we obtain

$$|\hat{f}_Q(\alpha)| \lesssim \alpha^{-\lambda} (\alpha(\alpha+2\lambda))^{-m} \int_1^\infty \frac{(t^2-1)^{\lambda-\frac{1}{2}} (2t)^\alpha}{(1+t^2)^{\alpha+1}} dt$$

$$\lesssim \alpha^{-(2m+\lambda)} \int_1^\infty \frac{(t^2-1)^{\lambda-\frac{1}{2}}}{t} dt \lesssim \alpha^{-(2m+\lambda)} \int_1^\infty t^{-\frac{3}{2}} (t-1)^{\lambda-\frac{1}{2}} dt$$

$$= \alpha^{-(2m+\lambda)} \int_0^\infty t^{\lambda-\frac{1}{2}} (t+1)^{\lambda-\frac{3}{2}} dt = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(1-\lambda)}{\Gamma\left(\frac{3}{2}\right)} \alpha^{-(2m+\lambda)}.$$

From this it follows that

$$\|\hat{f}_Q\|_{\infty,\lambda} = \sup_{1\le\alpha<\infty} |\hat{f}_Q(\alpha)| \le C < \infty. \tag{2.11}$$

On the other hand for $1 \le p < \infty$ we have

$$\|\hat{f}_Q\|_{p,\lambda} = \left(\int_1^\infty |\hat{f}_Q(\alpha)|^p (\alpha^2-1)^{\lambda-\frac{1}{2}} d\alpha\right)^{\frac{1}{p}}$$

$$\lesssim \left(\int_1^\infty \frac{(\alpha^2-1)^{\lambda-\frac{1}{2}}}{\alpha^{(2m+\lambda)p}} d\alpha\right)^{\frac{1}{p}} \lesssim \left(\int_1^\infty \frac{(\alpha^2-1)^{\lambda-\frac{1}{2}}}{\alpha^{2m+\lambda}} d\alpha\right)^{\frac{1}{p}}$$

$$\lesssim \left(\int_0^\infty \frac{\alpha^{\lambda-\frac{1}{2}}}{(\alpha+1)^{2m+\lambda}} d\alpha\right)^{\frac{1}{p}} = \left(\frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(2m)}{\Gamma\left(2m+\lambda+\frac{1}{2}\right)}\right)^{\frac{1}{p}} \le C < \infty.$$

From this and (2.1) it follows that $\hat{f}_Q \in L_{p,\lambda}$ for $f \in S$ and $1 \le p \le \infty$.

This shows that the transformations (2.1) and (2.3) also (2.2) and (2.4) are mutual inverse in S. According to Theorem 2 in [18], for every function f in $L_{1,\lambda}$, $F_P^{-1} F_P f = f$. On the other hand from the symmetry property of the operator A_{cht} and formula (1.7) we have

$$f(x) = C_\lambda \int_1^\infty \hat{f}_P(\alpha) Q_\alpha^\lambda(x) (\alpha^2-1)^{\lambda-\frac{1}{2}} d\alpha$$

$$= C_\lambda \int_1^\infty \left(\int_1^\infty f(t) P_\alpha^\lambda(t) (t^2-1)^{\lambda-\frac{1}{2}} dt\right) Q_\alpha^\lambda(x) (\alpha^2-1)^{\lambda-\frac{1}{2}} d\alpha$$

$$= \int_1^\infty \left(C_\lambda \int_1^\infty f(t) Q_\alpha^\lambda(t) (t^2-1)^{\lambda-\frac{1}{2}} dt\right) P_\alpha^\lambda(t) (\alpha^2-1)^{\lambda-\frac{1}{2}} d\alpha$$

$$= \int_1^\infty f(\alpha) P_\alpha^\lambda(x) (\alpha^2-1)^{\lambda-\frac{1}{2}} d\alpha,$$

that is, $F_P F_P^{-1} = f$.

Thus we obtain that

$$F_P F_P^{-1} f = f = F_P^{-1} F_P f$$

for $f \in L_{1,\lambda}$. Further taking into account (1.7) in (2.3) and (2.4) we obtain that $F_P^{-1} = F_Q^{-1} \in L_{1,\lambda}$. As in the case of F_P we can write that for any $f \in S$

$$F_Q F_Q^{-1} f = f = F_Q^{-1} F_Q f$$

and

$$F_P^{-1} F_P f = f = F_Q^{-1} F_Q f. \tag{2.12}$$

Since $S \subset L_{1,\lambda}$, from above results we confine that (2.1) and (2.3) and also (2.2) and (2.4) are mutual inverse in S.

Let S' be the set of continuous linear functionals on S, that is, S' is the space of temperate generalized functions. For $f \in S'$ and $\varphi \in S$ we denote by $f(\varphi) = < f, \varphi >$ the value of the functional f at φ. We denote the spaces $L_{p,\lambda}$, $1 \leq p \leq \infty$, in S' by putting

$$< f, \varphi > := \int_1^\infty f(t) \, \varphi(t) \, \left(t^2 - 1\right)^{\lambda - \frac{1}{2}} \, \mathrm{d}t \tag{2.13}$$

for $f \in L_{p,\lambda}$ and $\varphi \in S$.

The following Parseval equality holds for the Gegenbauer transformation (see [18], Lemma 8)

$$\int_1^\infty f(x) \, g(x) \, \left(x^2 - 1\right)^{\lambda - \frac{1}{2}} \, \mathrm{d}x = C_\lambda \int_1^\infty \hat{f}_P(\alpha) \, \hat{g}_Q(\alpha) \, \left(\alpha^2 - 1\right)^{\lambda - \frac{1}{2}} \, \mathrm{d}\alpha \tag{2.14}$$

for $f, g \in S$.

From this it follows that if $f \in S$, then take place the equality:

$$\int_1^\infty f^2(x) \, \left(x^2 - 1\right)^{\lambda - \frac{1}{2}} \, \mathrm{d}x = C_\lambda \int_1^\infty \hat{f}_P(\alpha) \, \hat{f}_Q(\alpha) \, \left(\alpha^2 - 1\right)^{\lambda - \frac{1}{2}} \, \mathrm{d}\alpha, \tag{2.15}$$

which we remind of the Parseval-Plancherel's formula.

Equality (2.14) can be written as

$$< f, g > = C_\lambda \, < F_P f, F_Q g > .$$

Taking into account (2.5), we transform last equality to form

$$< F_P f, g > = \frac{1}{C_\lambda} \, < f, F_Q^{-1} g > = < f, \overset{\vee}{g}_Q >, \tag{2.16}$$

where $\overset{\vee}{g}_Q = \frac{1}{C_\lambda} F_Q^{-1} g$ which follows from equality (2.12).

The Gegenbauer transformation can be extended to a class of functions wider than S.

We show that the Gegenbauer transformation $F_P f$ of generalized functions next are the generalized functions, i.e. of continuous linear functionals, which defined by properties:

1) $< F_P f, \varphi + \psi > = < F_P f, \varphi > + < F_P f \psi >, \quad \varphi, \psi \in S$,

2) $< F_P f, \alpha \varphi > = \alpha < F_P f, \varphi >, \quad \alpha \in \mathbb{R}$,

3) if $\varphi_n \to 0$, then $< F_P f, \varphi_n > \to 0$ at $n \to \infty$.

First two properties follow from the linearity property of the integrals. We will only prove the property 3).

The convergence to zero of the sequence $\varphi_n \in S$ is defined by the condition: for any natural numbers r and q and positive ε there exists a natural number such that whenever $n \geq n_0$ the inequality

$$\left(1 + x^2\right)^r \left| \varphi_n^{(q)}(x) \right| < \varepsilon \tag{2.17}$$

holds (see [10]).

Then we have

$$< F_P f, \varphi_n > = \int_1^\infty F_P f(\alpha) \, \varphi_n(\alpha) \, \left(\alpha^2 - 1\right)^{\lambda - \frac{1}{2}} \, \mathrm{d}\alpha$$

$$= \int_1^A F_P f(\alpha) \varphi_n(\alpha) \left(\alpha^2 - 1\right)^{\lambda - \frac{1}{2}} \, \mathrm{d}\alpha$$

$$+ \int_A^\infty F_P f(\alpha) \varphi_n(\alpha) \left(\alpha^2 - 1\right)^{\lambda - \frac{1}{2}} \, \mathrm{d}\alpha = J_1 + J_2. \tag{2.18}$$

Since (see [18]) $F_P f(\alpha) \to 0$ as $\alpha \to \infty$, we choose a number A so that for given $\varepsilon > 0$ the inequality

$$|F_P f(\alpha)| < \varepsilon \quad \text{for} \quad \alpha > A$$

holds.

Then

$$[J_2] < \varepsilon \int_A^\infty |\varphi_n(\alpha)| \left(\alpha^2 - 1\right)^{\lambda - \frac{1}{2}} \mathrm{d}\alpha. \tag{2.19}$$

As $\varphi_n \in S$, then by definition we have

$$|\varphi_n(\alpha)| \le \frac{c_r}{(1 + \alpha^2)^r}, \ r = 1, 2, \dots . \tag{2.20}$$

Taking into account (2.20) in (2.19) we obtain

$$|J_2| < c_r \ \varepsilon \int_A^\infty \frac{\left(\alpha^2 - 1\right)^{\lambda - \frac{1}{2}} \mathrm{d}\alpha}{(1 + \alpha^2)^r} = c_r \ \varepsilon \int_A^\infty \frac{(\alpha - 1)^{\lambda - \frac{1}{2}} \mathrm{d}\alpha}{(\alpha + 1)^{\frac{1}{2} - \lambda} (1 + \alpha^2)^r}$$

$$\le c_r \ \varepsilon \int_0^\infty \frac{(\alpha - 1)^{\lambda - \frac{1}{2}} \mathrm{d}\alpha}{\alpha^{2r + \frac{1}{2} - \lambda}} = c_r B\left(\lambda + \frac{1}{2}, 2r - 2\lambda\right) \ \varepsilon, \tag{2.21}$$

where $B(a, b)$ is of Euler integrals first type.

Since $\varphi_n(\alpha) \to 0$ as $n \to \infty$, by (2.17) we have

$$|\varphi_n(\alpha)| \le \frac{\varepsilon}{(1 + \alpha^2)^r}, \ r = 1, 2, \dots .$$

From here we have

$$|J_1| \le \varepsilon \int_1^A \frac{|F_P f(\alpha)| \ \left(\alpha^2 - 1\right)^{\lambda - \frac{1}{2}}}{(1 + \alpha^2)^r} \mathrm{d}\alpha.$$

Take into account (2.5), we obtain

$$|J_1| \le \varepsilon \int_1^A \frac{(\alpha - 1)^{\lambda - \frac{1}{2}} \mathrm{d}\alpha}{(1 + \alpha^2)^r \alpha^{1 - 2\lambda}} \le \ \varepsilon \ c_r \int_1^A (\alpha - 1)^{\lambda - \frac{1}{2}} \mathrm{d}\lambda$$

$$= c(\lambda, r) \ \varepsilon \ (\alpha - 1)^{\lambda + \frac{1}{2}} |_1^A = c(\lambda, r)(A - 1)^{\lambda + \frac{1}{2}} \ \varepsilon. \tag{2.22}$$

Taking into account (2.21) and (2.22) in (2.18), we obtain

$$< F_P f, \varphi_n > \to 0 \text{ as } n \to \infty.$$

In this way, the Gegenbauer transformation $F_P f$ for $f \in S'$ next is the generalized function, i.e. $F_P f \in S'$. Therefore the formula (2.16) can be naturally extended on functions from S', i.e., for $f \in S'$ by definition

$$< F_P f, g > := \frac{1}{C_\lambda} < f, F_P^{-1} g >, \quad g \in S. \tag{2.23}$$

We note an important property of Gegenbauer transformation (see [14], Lemma 3.3): if $f \in S'$, then the following inequality holds:

$$\left(\widehat{D_\lambda^k f}\right)_P (\alpha) = (\alpha(\alpha + 2\lambda))^k \hat{f}_P(\alpha), \quad k = 1, 2 \dots . \tag{2.24}$$

In the class S, the operator D_λ is symmetric in the following sence (see [14], Lemma 4), i.e.,

$$< D_\lambda \psi, \varphi > = < \psi, D_\lambda \varphi >, \quad \varphi, \psi \in S,$$

that is equivalent

$$\int_0^\infty (D_\lambda \psi)(ch \ x) \varphi(ch \ x) sh^{2\lambda} x \mathrm{d}x = \int_0^\infty \psi(ch \ x)(D_\lambda \varphi)(ch \ x) sh^{2\lambda} x \mathrm{d}x.$$

Reflections

$$\varphi \mapsto \int_0^\infty (D_\lambda \psi)(ch \ x) \ \varphi(ch \ x) \ sh^{2\lambda} x \mathrm{d}x$$

and

$$f \mapsto \int_0^\infty \psi(ch\ x) f\left(ch\ x\right) sh^{\,2\lambda} x \mathrm{d}x$$

define the continuous linear functionals on S. Denoting their over $D_\lambda\,\psi$ and ψ, we rewrite the last equality by form

$$\left(D_\lambda\,\psi\right)(\varphi) = \psi\left(D_\lambda\,\varphi\right),$$

or the same

$$< D_\lambda\,\psi, \varphi > = < \psi, D_\lambda, \varphi >, \quad \psi \in S', \quad \varphi \in S. \tag{2.25}$$

Since, right-hand side of (2.25) is defined for all $\psi \in S'$ and $\varphi \in S$, and representation $\varphi \to \psi\left(D_\lambda\,\varphi\right)$ is continuous on S, as superposition of two continuous functions, then we can define $D_\lambda\,\psi$ of generalized function $\psi \in S'$ by means of (2.25).

Obviously, that then $D_\lambda \psi \in S'$.

From inequalities (1.10), (2.5) and also of the self-adjoind of the operator $A^\lambda_{ch\ t}$ follows that for $\varphi \in L_{1,\lambda}$ the equality holds:

$$< A^\lambda_{ch\ t}\,\hat{f}_P, \varphi > = < \hat{f}_P, A^\lambda_{ch\ t}\,\varphi > .$$

Since $\hat{f}_P(\alpha) \in S'$, then naturally extended the generalized Gegenbauer shift operator on generalized functions space by formula

$$< A^\lambda_{ch\ t} f, \varphi > := < f, A^\lambda_{ch\ t}\,\varphi >, \ f \in S', \ \varphi \in S. \tag{2.26}$$

Let a be arbitrary positive number. The dilation operator (see [16])

$$f_a\left(ch\ x\right) = \frac{f\left(ch\ \frac{x}{a}\right) sh^{2\lambda}\frac{x}{a}}{a\ sh^{2\lambda}\,x}, \ \ a > 0 \tag{2.27}$$

of the function in $L_{1,\,\lambda}$ preserve meaning of the integral, that is

$$\int_0^\infty f_a\left(ch\ x\right) sh^{2\lambda}\,x\mathrm{d}x = \int_0^\infty f\left(ch\ x\right) sh^{2\lambda}\,x\mathrm{d}x. \tag{2.28}$$

Really,

$$\int_0^\infty f_a\left(ch\ x\right) sh^{2\lambda}\,x\mathrm{d}x = \int_0^\infty \frac{f\left(ch\ \frac{x}{a}\right) sh^{2\lambda}\frac{x}{a}}{a\ sh^{2\lambda}\,x} sh^{2\lambda}\,x\mathrm{d}x$$

$$= \int_0^\infty f\left(ch\ \frac{x}{a}\right) sh^{2\lambda}\,\frac{x}{a}d\frac{x}{a} = \int_0^\infty f\left(ch\ x\right) sh^{2\lambda}\,x\mathrm{d}x.$$

From (2.29) it follows that

$$\|f_a\|_{1,\,\lambda} = \|f\|_{1,\,\lambda}. \tag{2.29}$$

The following correlations is connect of the generalized Gegenbauer shift operator and of Gegenbauer transformation (see [18], Lemma 1):

$$\left(\widehat{A_{ch\ t}f}\right)_P (\alpha) = \hat{f}_P(\alpha)\,Q^\lambda_\alpha(ch\,t), \ \ \alpha \in [1,\infty), \ \ t \in [0,\infty). \tag{2.30}$$

The operators $A_{ch\ t}$ and D_λ commute, that is,

$$\left(D_\lambda A_{ch\ t}f\right)(ch\ x) = \left(A_{ch\ t}D_\lambda f\right)(ch\ x), \quad f \in S. \tag{2.31}$$

This equality is a particular case of more qeneral equality (see further Lemma 3.4)

$$\left(A^k_{ch\ t}D^r_\lambda f\right)(ch\ x) = \left(D^r_\lambda A_{ch\ t}f\right)(ch\ x), \quad k, r = 1, 2, ..., \quad f \in S, \tag{2.32}$$

although it can be proved by application of the Gegenbauer transformation of both side of (2.32). Taking into account (2.6) and (2.30), we have

$$\left(\widehat{D_\lambda A_{ch\,t}f}\right)_P (\alpha) = \alpha\left(\alpha + 2\lambda\right)\left(\widehat{A_{ch\ t}f}\right)_P (\alpha) = \alpha\left(\alpha + 2\lambda\right) Q^\lambda_\alpha(ch\,t)\hat{f}_P(\alpha)$$

$$= \hat{f}_P(\alpha)D_\lambda Q_\alpha^\lambda(ch\,t) = \left(\widehat{A_{ch\,t}D_\lambda f}\right)_P(\alpha),$$

from this (2.32) follows.

The convolution of functions f and g on $\mathbb{R}_+$ is defined by the relation

$$(f * g)(ch\,x) = \int_0^\infty g(ch\,t)A_{ch\,t}f(ch\,x)\,sh^{\,2\lambda}\,tdt. \tag{2.33}$$

The convolution has sense if the integral on the right-hand side of (2.28) is convergence. If $f \in L_{p,\lambda}$, $g \in L_{1,\lambda}$, then the convolution is defined and belong $L_{p,\lambda}$, $1 \le p \le \infty$, as (see [18], Lemma 4)

$$\|f * g\|_{p,\lambda} \le \|f\|_{p,\lambda} \cdot \|g\|_{1,\lambda}. \tag{2.34}$$

From here, in particular it follows that for $f, g \in S$ their convolution $f * g \in S$.

We deriv some properties of the convolution.

1. The convolution is commutative, that is,

$$f * g = g * f. \tag{2.35}$$

2.

$$\left(\widehat{f * g}\right)_P(\alpha) = \hat{f}_P(\alpha)\hat{g}_Q(\alpha), \tag{2.36}$$

from this and also (2.12) it follows that

3.

$$f * g = F_P^{-1}\left(\hat{f}_P\,\hat{g}_Q\right). \tag{2.37}$$

4.

$$D_\lambda^r(f * g) = (D_\lambda^r f) * g = f * (D_\lambda^r g), \qquad r = 1, 2..., \tag{2.38}$$

which follows from (2.36) by using (2.6).

Indeed,

$$\left(\widehat{D_\lambda^r f * g}\right)_P(\alpha) = \left(\widehat{D_\lambda^r f}\right)_P(\alpha)\,\hat{g}_Q(\alpha) = (\alpha(\alpha + 2\lambda))\,\hat{f}_P(\alpha)\hat{g}_Q(\alpha)$$

$$= (\alpha(\alpha + 2\lambda))^r\left(\widehat{f * g}\right)_P(\alpha) = \left(\widehat{D_\lambda^r(f * g)}\right)_P(\alpha).$$

From this and (2.35) it follows that (2.38).

5. It is associative, that is,

$$(f * g) * h = f * (g * h).$$

Indeed, by using commutativity of the operator $A_{ch\,t}$, we obtain

$$((f * g) * g)(ch\,x) = \int_0^\infty (f * g)(ch\,t)A_{ch\,t}h(ch\,x)\,sh^{\,2\lambda}\,tdt$$

$$= \int_0^\infty \left(\int_0^\infty f(ch\,x)A_{ch\,x}g(ch\,t)sh^{\,2\lambda}\,xdx\right)A_{ch\,t}h(ch\,x)\,sh^{\,2\lambda}\,tdt$$

$$= \int_0^\infty \int_0^\infty g(ch\,x)A_{ch\,x}f(ch\,t)A_{ch\,t}h(ch\,x)\,sh^{\,2\lambda}\,x\,sh^{\,2\lambda}\,tdxdt$$

$$= \int_0^\infty \int_0^\infty g(ch\,x)A_{ch\,t}f(ch\,x)A_{ch\,x}h(ch\,t)sh^{\,2\lambda}\,xsh^{\,2\lambda}\,tdxdt$$

$$= \int_0^\infty (g * h)(ch\,t)A_{ch\,t}f(ch\,x)\,sh^{\,2\lambda}\,tdt = (f * (g * h))(ch\,x).$$

These properties hold for $f, g, h \in S$.

In case then $f \in S'$ and $g \in S$ the convolution $f * g$ is defined by formula

$$(f * g)(ch\,x) := < f, A_{ch\,x}g > = f(A_{ch\,x}g), \quad x \in \mathbb{R}_+. \tag{2.39}$$

The right-hand of (2.39) exist, so far as f is slow growing function, and $A_{ch\ x}g \in S$ on the strength inequality (1.10). Moreover the reflection $g \mapsto f(A_{ch\ x}g)$ is continuous as superposition two continuous functions.

Theorem 2.1. *Let $f \in S'$ and $g \in S$. Then their convolution $u(ch\ x) = f(A_{ch\ x}g)$ are generalized function slow growing, i.e. $u \in S'$.*

Proof. First we show, that $u \in C^\infty$. Using (2.33), we can write

$$D_\lambda^k u(ch\ x) = \int_0^\infty f(ch\ t) A_{ch\ x} D_\lambda^k g(ch\ t) sh^{\,2\lambda}\ tdt = f\left(A_{ch\ x} D_\lambda^k g\right),\ k = 1, 2, \ldots. \qquad (2.40)$$

As, $g \in S$, then $D_\lambda^k g$ exists and continuity for any $k \in \mathbb{N}$. But then on the strength of continuous of the operator $A_{ch\ x}^\lambda$ is continuous and the right-hand side (2.40), as superposition two continuous functions. In this way $u \in C^\infty$.

We show that u is slow growing function. Taking into account of Taylor-Delsart formula (see [14], Lemma 6), and also of Lemma 3.6 (see further), we obtain

$$A_{ch\ t}^\lambda f(ch\ x) = \sum_{\nu=0}^{n-1} C_\nu(ch\ t) D_\lambda^\nu f(ch\ x) + C_n(ch\ t) A_{ch\ t}^\lambda D_\lambda^n f(ch\ x).$$

From here and from (2.39) we have

$$u(ch\ x) = \int_0^\infty g(ch\ t) A_{ch\ t}^\lambda f(ch\ x)\ sh^{\,2\lambda}\ tdt$$

$$= \sum_{\nu=0}^{n-1} D_\lambda^\nu f(ch\ x) \int_0^\infty C_\nu(ch\ t) g(ch\ t)\ sh^{\,2\lambda}\ tdt$$

$$+ \int_0^\infty C_n(ch\ t)\ A_{ch\ t}\ D^n f(ch\ x)\ g(ch\ t)\ sh^{\,2\lambda}\ tdt. \qquad (2.41)$$

Since

$$ch\ (x - t) \leq ch\ x\ ch\ t - sh\ x\ sh\ t\ \cos\ \varphi \leq ch\ (x + t),$$

then

$$\left| A_{ch\ t}^\lambda f(ch\ x) \right| \leq \sup_{1 \leq u \leq ch\ (x+t)} |f(u)|.$$

But then

$$\left| \int_0^\infty C_n(ch\ t) A_{ch\ t} D_\lambda^n f(ch\ x)\ g(ch\ t) sh^{\,2\lambda}\ tdt \right|$$

$$\leq \sup_{1 \leq u \leq ch\ (x+t)} |D_\lambda^n f(u)| \int_0^\infty C_k(ch\ t)\ |g(ch\ t)| sh^{\,2\lambda}\ tdt. \qquad (2.42)$$

Further, since (see [14], Lemma 5)

$$C_k(ch\ t) \leq c(k, \lambda)(ch\ t - 1)^k,\ k = 1, 2, \ldots,$$

but $g \in S$, that the integral

$$\int_0^\infty C_k(ch\ t)\ |g(ch\ t)|\ sh^{\,2\lambda}\ tdt$$

exist. But then from (2.42) and (2.41) it follows that

$$u(ch\ x) \lesssim \sum_{\nu=0}^n \sup_{1 \leq u \leq ch(x+t)} D_\lambda^\nu f(u). \qquad (2.43)$$

Since (see [14], Lemma 1)

$$D_\lambda f(x) = (x^2 - 1) f''(x) + (2\lambda + 1) x f'(x),$$

it follows that the convolution $u(ch\, x)$ has growth polynomials from (2.43), i.e., it is a slow growing function.

Thus the theorem is proved.

Now let $f, g, h \in S$. Using the Fubini theorem we show that

$$(f * g) * (h) = < f * g, h >$$

$$= \int_0^\infty \left(\int_0^\infty A_{ch\,t}^\lambda f(ch\, x)\, g(ch\,t)\, sh^{2\lambda}\, t \mathrm{d}t \right) h(ch\, x)\, sh^{2\lambda}\, x \mathrm{d}x$$

$$= \int_0^\infty \left(\int_0^\infty A_{ch\,t}^\lambda f(ch\, x)\, h(ch\, x)\, sh^{2\lambda}\, x \mathrm{d}x \right) g(ch\,t)\, sh^{2\lambda}\, t \mathrm{d}t$$

$$= \int_0^\infty \left(\int_0^\infty f(ch\, x)\, A_{ch\,t}^\lambda h(ch\, x)\, sh^{2\lambda}\, x \mathrm{d}x \right) g(ch\,t)\, sh^{2\lambda}\, t \mathrm{d}t$$

$$= < g, f * h > = < g * f, h > = < f, g * h >,$$

i.e.,

$$(f * g) * (h) = f * (g * h).$$

If $f \in S'$ and $g, h \in S$, then right-hand side of the last equality is defined since $g * h \in S$. Moreover, the representation $h \mapsto f(g * h)$ is continuous which is the superposition of two functions. In this way the convolution of the generalized function f and a basic function g can be defined by the following equality

$$< f * g, h > = < f, g * h >, \; f \in S', \quad g, h \in S. \tag{2.44}$$

We show that for all $f \in S'$ and $g, h \in S$ the convolution is associative, that is,

$$(f * g) * h = f * (g * h).$$

Indeed, if $f \in S'$, and $g, h \in S$, then

$$(f * g) * h(ch\, x) = \int_0^\infty (f * g)(ch\,t)\, A_{ch\,t}^\lambda h(ch\, x)\, sh^{2\lambda}\, t \mathrm{d}t$$

$$= \int_0^\infty \left(\int_0^\infty f(ch\, x)\, A_{ch\,x}^\lambda g(ch\,t) sh^{2\lambda}\, x \mathrm{d}x \right) A_{ch\,t}^\lambda h(ch\, x)\, sh^{2\lambda}\, t \mathrm{d}t$$

$$= \int_0^\infty \int_0^\infty g(ch\, x)\, A_{ch\,x}^\lambda f(ch\,t) A_{ch\,x}^\lambda (ch\, x)\, sh^{2\lambda}\, x sh^{2\lambda}\, t \mathrm{d}x \mathrm{d}t$$

$$= \int_0^\infty \int_0^\infty g(ch\, x)\, A_{ch\,x}^\lambda f(ch\, x)\, A_{ch\,x}^\lambda (ch\,t) sh^{2\lambda}\, x sh^{2\lambda}\, t \mathrm{d}x \mathrm{d}t$$

$$= \int_0^\infty (g * h)(ch\,t) A_{ch\,t}^\lambda f(ch\, x)\, sh^{2\lambda}\, t \mathrm{d}t = f * (g * h)(ch\, x). \tag{2.45}$$

Since $g, h \in S$, that their convolution $g * h$ belongs to S, but the right-hand part of (2.45) according to (2.44) it is convolution of the shift function $f \in S'$ and the test function $g * h \in S$.

We show that the convolution of $f \in S'$ and $g \in S$ is defined by formula (2.39). It remains to show that the properties (2.35), (2.36) and (2.38) are provided.

Indeed, using (2.26) and (2.39) we obtain

$$(f * g)(ch\, x) = < f, A_{ch\,t}^\lambda g > = < A_{ch\,t}^\lambda f, g > = (g * f)(ch\, x). \tag{2.46}$$

Further, using equalities (2.46), and also (1.7) and by changing the order of integration we obtain

$$\left(\widehat{f * g}\right)_P (\alpha) = \int_0^\infty (f * g)\,(ch\ x)\,P_\alpha^\lambda\,(ch\ x)\,sh^{\,2\lambda}\,x\mathrm{d}x$$

$$= \int_0^\infty \left(\int_0^\infty g(ch\,t)A_{ch\ t}f\,(ch\ x)\,sh^{\,2\lambda}\,t\mathrm{d}t\right)P_a^\lambda\,(ch\ x)\,sh^{\,2\lambda}\,x\mathrm{d}x$$

$$= \int_0^\infty \left(\int_0^\infty f(ch\,t)A_{ch\ t}P_a^\lambda\,(ch\ x)\,sh^{\,2\lambda}\,t\mathrm{d}t\right)g\,(ch\ x)\,sh^{\,2\lambda}\,x\mathrm{d}x$$

$$= \int_0^\infty \left(\int_0^\infty f(ch\,t)P_\alpha^\lambda(ch\,t)sh^{\,2\lambda}\,t\mathrm{d}t\right)g\,(ch\ x)\,Q_\alpha^\lambda\,(ch\ x)\,sh^{\,2\lambda}\,x\mathrm{d}x = \hat{f}_P(\alpha)\hat{g}_Q(\alpha),$$

i.e., take place (2.36).

Using consistent of the formulas (2.40), (2.39), (2.25), (2.32) and (2.26), we obtain

$$D_\lambda^r (f * g) = f\,(A_{ch\ x}D_\lambda^r g) =< f, A_{ch\ x}D_\lambda^r g >= f * (D_\lambda^r g) =< A_{ch\ x}f, D_\lambda^r g >$$

$$=< D_\lambda^r A_{ch\ x}f, g >=< A_{ch\ x}D_\lambda^r f, g >=< D_\lambda^r f, A_{ch\ x}g >= (D_\lambda^r f) * g,$$

i.e., take place (2.38).

3   Auxiliary Results

For the proof of Theorems 1.1 and 1.2 we need some auxiliary assertions.

We consider the function

$$C_k(chx) = -\int\limits_1^{chx} \theta\,(chx, \sigma)\,C_{k-1}(\sigma)\,\left(\sigma^2 - 1\right)^{\lambda - \frac{1}{2}}\,d\sigma\ ,\ \ C_0 = 1,\ \ k = 1, 2, \ldots,$$

where

$$\theta\,(chx, \sigma) = \begin{cases} -\int\limits_\sigma^{chx} \left(u^2 - 1\right)^{-\lambda - \frac{1}{2}}\,du,\ \ 1 < \sigma < chx\,, \\ 0,\qquad\qquad\text{for the other}\qquad \sigma\,, \end{cases}$$

$$R_1(chs)f(chx) = \int_1^{chs} \theta\,(chs, \sigma)\,(A_\sigma D_\lambda f)\,(chx)\,\left(\sigma^2 - 1\right)^{\lambda - \frac{1}{2}}\,d\sigma,$$

and

$$R_k(chs)f(chx) = \int_1^{chs} \theta\,(chs, \sigma)\,(R_{k-1}(\sigma)D_\lambda f)\,(chx)\,\left(\sigma^2 - 1\right)^{\lambda - \frac{1}{2}}\,d\sigma,$$

$$k = 2, 3, \ldots \qquad\qquad (3.1)$$

The following assertion is just (see [13]):

Lemma 3.1. [14] If $f \in D_\lambda^{n-1}[0, \infty)$, then the Taylor-Delsarte's formula is valid:

$$R_n(chs)f(chx) = A_{cht}f(chx) - \sum_{k=0}^{n-1} C_k\,(ch\,s)\,D_\lambda^k f(chx),\ \ n = 1, 2, \ldots,$$

$$D_\lambda^0 f\,(ch\,x) \equiv f\,(ch\,x)\,.$$

Lemma 3.2. Let $f \in L_{p,\lambda}$. Then the following equality is valid:

$$\left(\widehat{A_{cht}^k f}\right)_P (\alpha) = \hat{f}_P(\alpha)\,\left(Q_\alpha^\lambda(cht)\right)^k\ ,\ \ k = 1, 2, \ldots\ .$$

Lemma 3.3. Let $f \in W_{p,\lambda}^k$, then

$$\left(\widehat{D_\lambda^k f}\right)_P(\alpha) = (\alpha(\alpha+2\lambda))^k \widehat{f}_P(\alpha), \quad k = 1, 2, \dots.$$

Lemma 3.4. If $f \in S$, then following equality is valid:

$$\left(A_{cht}^k D_\lambda^r f\right)(chx) = \left(D_\lambda^r A_{cht}^k f\right)(chx), \quad k, r = 1, 2, \dots.$$

Lemma 3.5. Following equality is valid:

$$A_{cht}^k R_n(chs) f(chx) = R_n(chs) \left(A_{chs}^k f\right)(chx), \quad k = 1, 2, \dots.$$

Lemma 3.6. The following equality is valid:

$$R_n(chs) f(chx) = C_n(chs) A_{chs} D_\lambda^n f(chx), \quad n = 1, 2, \dots.$$

Lemma 3.7. If $f \in D_\lambda^k[0, \infty)$, then the following equality is valid:

$$\Delta_{ch\,s}^k f(ch\,x) = R_k(ch\,s)\left(A_{ch\,s}^{k-1} f\right)(ch\,x), \quad k = 1, 2, \dots.$$

Lemma 3.8. If the function $f \in D_\lambda^k[0, \infty)$, then the following equality is valid:

$$\Delta_{ch\,s}^k f(ch\,x) = C_k(ch\,s) A_{ch\,s}^k \left(D_\lambda^k f\right)(ch\,x), \quad k = 1, 2, \dots. \tag{3.2}$$

Lemma 3.9. If $f \in D_\lambda^k[0, \infty)$, then the following equality is valid:

$$\Delta_{ch\,t}^m D_\lambda^k f(ch\,x) = D_\lambda^k \Delta_{ch\,t}^m f(ch\,x), \quad k = 1, 2, \dots; \quad m = 1, 2, \dots.$$

Lemma 3.10. *For any function $f \in W_{p,\lambda}^k$, $k = 1, 2, \dots$, $1 \le p \le \infty$ the following inequality is valid:*

$$\left\|\Delta_{ch\,t}^k f\right\|_{p,\lambda} \le \frac{\Gamma(2\lambda+1)}{\Gamma(k+1)\Gamma(2k+2\lambda-1)}(ch\,t-1)^k \left\|D_\lambda^k f\right\|_{p,\lambda}. \tag{3.3}$$

Proof. Since $f \in W_{p,\lambda}^k$, then from of the equality (3.2) it follows that

$$\left\|\Delta_{ch\,t}^k f\right\|_{p,\lambda} \le C_k(ch\,t) \left\|\left(A_{ch\,t}^k D_\lambda^k\right) f\right\|_{p,\lambda}.$$

Taking into account (1.10) and also the inequality (see [14], Lemma 5)

$$C_k(ch\,t) \le \frac{\Gamma(2\lambda+1)}{\Gamma(k+1)\Gamma(2k+2\lambda-1)}(ch\,t-1)^k, \tag{3.4}$$

we obtain

$$\left\|\Delta_{ch\,t}^k f\right\|_{p,\lambda} \le C_k(ch\,t) \left\|D_\lambda^k f\right\|_{p,\lambda} \le \frac{\Gamma(2\lambda+1)}{\Gamma(k+1)\Gamma(2k+2\lambda-1)}(ch\,t-1)^k \left\|D_\lambda^k f\right\|_{p,\lambda}.$$

Thus Lemma 3.10 is proved.

Lemma 3.11. *Let $f \in L_{1,\lambda}$, $g \in W_{p,\lambda}^r$, $1 \le p \le \infty$. If f is the spectrum of order ν, then their convolution is the spectrum of order ν and belongs to the class $\mathfrak{M}(\nu, p, \lambda)$.*

Theorem 3.1. *For $f \in W_{p,\lambda}^r$, $1 \le p \le \infty$ the following inequality is valid*

$$E_\nu(f)_{p,\lambda} \lesssim \nu^{-2r} \left\|D_\lambda^r f\right\|_{p,\lambda}. \tag{3.5}$$

Proof. We can write difference (1.11) in a form

$$-\Delta_{ch\,t}^k f(ch\,x) = -\sum_{i=0}^k (-1)^{k-i} \binom{k}{i} A_{ch\,t}^i f(ch\,x)$$

$$= \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} A^i_{ch\,t} f\,(ch\,x) - f\,(ch\,x)\,. \tag{3.6}$$

Let $\varphi\,(ch\,t)$ be the spectrum of order ν satisfied the condition

$$\int_0^\infty \varphi\,(ch\,t)\,sh^{\,2\lambda}\,t dt = 1, \tag{3.7}$$

i.e., $\varphi \in \mathfrak{M}\,(1, 1, \lambda)$.

We consider the function

$$\Phi_\nu\,(ch\,x) = \int_0^\infty \varphi\,(ch\,t) \left(-\Delta^k_{ch\,\frac{t}{\nu}} f\,(ch\,x) + f\,(ch\,x) \right) sh^{\,2\lambda}\,t dt$$

$$= \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} \int_0^\infty \varphi\,(ch\,t)\,A^i_{ch\,\frac{t}{\nu}} f\,(ch\,x)\,sh^{\,2\lambda}\,t dt. \tag{3.8}$$

Since the operator $A_{ch\,t}$ is self-adjoint, we have

$$\int_0^\infty \varphi\,(ch\,t)\,A^i_{ch\,\frac{t}{\nu}} f\,(ch\,x)\,sh^{\,2\lambda}\,t dt = \nu \int_0^\infty \varphi\,(ch\,\nu t)\,A^i_{ch\,t} f\,(ch\,x)\,sh^{\,2\lambda}\,\nu t dt$$

$$= \int_0^\infty A_{ch\,t} \left(A^{i-1}_{ch\,t} f\,(ch\,x) \right) \left(\frac{\nu\,\varphi\,(ch\,\nu t)\,sh^{\,2\lambda}\,\nu t}{sh^{\,2\lambda}\,t} \right) sh^{\,2\lambda}\,t dt.$$

Taking into account commutativity of the operator $A_{ch\,t}$

$$A_{ch\,t}\,A_{ch\,x} = A_{ch\,x}\,A_{ch\,t},$$

we obtain

$$\int_0^\infty A_{ch\,t} \left(A^{i-1}_{ch\,x}\,\varphi_{\frac{1}{\nu}}\,(ch\,t) \right) f\,(ch\,t)\,sh^{\,2\lambda}\,t dt$$

$$= \int_0^\infty A_{ch\,x} \left(A^{i-1}_{ch\,t}\,\varphi_{\frac{1}{\nu}} \right) (ch\,t)\,f\,(ch\,t)\,sh^{\,2\lambda}\,t dt.$$

Then from (3.8) we have

$$\Phi_\nu\,(ch\,x) = (K_\nu * f)\,(ch\,x) = \int_0^\infty (A_{ch\,x} K_\nu)\,(ch\,t)\,f\,(ch\,t)\,sh^{\,2\lambda}\,t dt, \tag{3.9}$$

where

$$K_\nu\,(ch\,t) = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} \left(A^{i-1}_{ch\,t}\,\varphi_{\frac{1}{\nu}}\,(ch\,t) \right). \tag{3.10}$$

Taking into account that $\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} = 1$, inequality (1.10) and (3.10) we obtain

$$\|K_\nu\|_{1,\lambda} \leq \left\| \varphi_{\frac{1}{\nu}} \right\|_{1,\lambda} = \|\varphi\|_{1,\lambda} = 1,$$

while the dilation operator $\varphi_{\frac{1}{\nu}}$ is an automorphism of $L_{1,\lambda}$.

Since $K_\nu \in L_{1,\lambda}$, by Lemma 3.11 the operator Φ_ν is of the spectrum of order ν, i.e., $\Phi_\nu \in \mathfrak{M}\,(p, \nu, \lambda)$, moreover is an entire function of exponential type (see proof of the Lemma 3.11).

Let $f \in W^r_{p,\lambda}$. Using Lemma 3.10 and taking into account (3.7) and (3.8), we obtain

$$\|\Phi_\nu f - f\|_{p,\lambda} \leq c\,(\lambda, r) \int_0^\infty \left\| \Delta^r_{ch\,\frac{t}{\nu}} f\,(ch(\cdot)) \right\|_{p,\lambda} \varphi\,(ch\,t)\,sh^{\,2\lambda}\,t dt$$

$$\leq c\left(\lambda,r\right)\left\|D_\lambda^r f\right\|_{p,\lambda}\int_0^\infty \left(ch\,\frac{t}{\nu}-1\right)^r \varphi\left(ch\,t\right)sh^{\,2\lambda}\,t\mathrm{d}t$$

$$\leq c(\lambda,r)\left\|D_\lambda^r f\right\|_{p,\lambda}\int_0^\infty \left(sh\,\frac{t}{2\nu}\right)^{2r} \varphi\left(ch\,t\right)sh^{\,2\lambda}\,t\mathrm{d}t$$

$$\leq c(\lambda,r)\nu^{-2r}\left\|D_\lambda^r f\right\|_{p,\lambda}\int_0^\infty \left(sh\,t\right)^{2r+2\lambda}\varphi\left(ch\,t\right)\mathrm{d}t. \tag{3.11}$$

In the capacity of the function $\varphi\left(ch\,t\right)$ we take the function

$$\varphi\left(ch\,t\right)=\frac{2}{B\left(\lambda+\frac{1}{2},\,r+\frac{1}{2}\right)}\left(ch\,t\right)^{-2\lambda-2r-1}.$$

Then (see [11], p. 361) using of formula

$$\int_0^\infty \frac{\left(sh\,x\right)^{2\mu-1}ch\,x\,\mathrm{d}x}{\left(1+a\,sh^{\,2}x\right)^\nu}=\frac{1}{2}a^{-\mu}B\left(\mu,\nu-\mu\right),\ \left(\nu>\mu>0\right),\,a>0 \tag{3.12}$$

at $\mu=\lambda+\dfrac{1}{2}$, $\nu=r+\lambda+1$ and $a=1$, we obtain

$$\int_0^\infty \varphi\left(ch\,t\right)sh^{\,2\lambda}\,t\mathrm{d}t=\frac{2}{B\left(\lambda+\frac{1}{2},r+\frac{1}{2}\right)}\int_0^\infty \frac{sh^{\,2\lambda}\,t\,ch\,t\mathrm{d}t}{\left(ch\,t\right)^{2r+2\lambda+2}}=1.$$

On the other hand, taking in the formula (3.12) $\mu=r+\lambda+\dfrac{1}{2}$, $\nu=r+\lambda+1$ and $a=1$ we obtain

$$\int\limits_0^\infty \varphi\left(ch\,t\right)\left(sh\,t\right)^{2\lambda+2r}\mathrm{d}t=\int\limits_0^\infty \frac{\left(sh\,t\right)^{2\lambda+2r}ch\,t\mathrm{d}t}{\left(ch\,t\right)^{2r+2\lambda+2}}=\frac{1}{2}B\left(r+\lambda+\frac{1}{2},\frac{1}{2}\right). \tag{3.13}$$

Taking into account (3.13) in (3.11), we have

$$\left\|\varPhi_\nu f-f\right\|_{p,\lambda}\leq c(\lambda,r)\nu^{-2r}\left\|D_\lambda^r f\right\|_{p,\lambda},$$

from this the assertion of Theorem 3.12 follows.

Let $K\left(f,t;L_{p,\lambda};W_{p,\lambda}^m\right)$ be K- Peetre functional constructing on the spaces $L_{p,\lambda}$ and $W_{p,\lambda}^m$, $f\in L_{p,\lambda}$, $t>0$. For brevity we take

$$K_m\left(f,t\right)_{p,\lambda}:=K\left(f,t^{2m};L_{p,\lambda};W_{p,\lambda}^m\right). \tag{3.14}$$

The modulus $\omega_k\left(f,t\right)_{p,\lambda}$ of smoothness of order k is defined on Section 1 by the formula (1.12)

$$\omega_k\left(f,t\right)_{p,\lambda}=\sup_{0<h\leq t}\left\|\varDelta_{ch\,h}^k f\right\|_{p,\lambda}.$$

Proposition 3.1. The modulus of smoothness $\omega_k\left(f,t\right)_{p,\lambda}$, $k\in\mathbb{N}$, $1\leq p\leq\infty$ has the following properties:

1. $\omega_k\left(f,t\right)_{p,\lambda}$ is a non-decreasing function of $t\in[0,\infty)$,
2. $\omega_k\left(f+g,t\right)_{p,\lambda}\leq\omega_k\left(f,t\right)_{p,\lambda}+\omega_k\left(g,t\right)_{p,\lambda}$,
3. $\omega_k\left(f,t\right)_{p,\lambda}\leq 2^k\left\|f\right\|_{p,\lambda}$,
4. if $f\in L_{p,\lambda}$, $1\leq p\leq\infty$, then $\lim\limits_{t\to 0}\omega_k\left(f,t\right)_{p,\lambda}=0$,
5. if $\nu\leq k$, then $\omega_k\left(f,t\right)_{p,\lambda}\leq 2^{k-\nu}\omega_\nu\left(f,t\right)_{p,\lambda}$,
6. if $f\in W_{p,\lambda}^m$, then for any $k>m$ the following inequality is valid:

$$\omega_k\left(f,t\right)_{p,\lambda}\lesssim t^{2m}\,\omega_{k-m}\left(D_\lambda^m f,t\right)_{p,\lambda},$$

7. if $f \in W_{p,\lambda}^{k}$, then the inequality is valid:

$$\omega_k \left(f,t\right)_{p,\lambda} \le 2^k \left(sh\,\frac{t}{2}\right)^{2k} \left\|D_\lambda^k f\right\|_{p,\lambda}.$$

We define a family of the operators P_ν^n with $\nu > 0$ by the formula $P_v'f = A_{ch\,\frac{1}{v}}'f$,

$$P_\nu^n f \equiv P_\nu f = A_{ch\,\frac{1}{\nu}}^n f - \sum_{k=1}^{n-1} C_k \left(ch\,\frac{1}{\nu}\right)\left(A_{ch\,\frac{1}{\nu}}^{n-1} D_\lambda^k f\right), n = 2, 3, \cdots. \tag{3.15}$$

Proposition 3.2. *Let $f \in \mathfrak{M}(\nu,p,\lambda)$, $1 \le p \le \infty$, $\nu > 0$. Then $P_\nu f \in \mathfrak{M}(\nu,p,\lambda)$.*
Proposition 3.3. *Let $f \in L_{p,\lambda}$, $1 \le p \le \infty$, $\nu > 0$. For every $t \in \left(0,\frac{1}{\nu}\right)$ the following inequalities hold:*

$$\left\|D_\lambda^m P_\nu\left(f\right)\right\|_{p,\lambda} \lesssim \left(sh\,\frac{t}{2}\right)^{-2m} \left\|\varDelta_{ch\,t}^m f\right\|_{p,\,\lambda}, \tag{3.16}$$

and if $f \in W_{p,\lambda}^m$, then

$$\left\|\varDelta_{ch\,t}^m P_\nu\left(f\right)\right\|_{p,\lambda} \lesssim \left(sh\,\frac{t}{2}\right)^{2m} \left\|D_\lambda^m f\right\|_{p,\lambda}. \tag{3.17}$$

Proof. Using the equality (3.2) and also Lemmas 3.2 and 3.3, we obtain

$$\left(\widehat{\varDelta_{ch\,t}^m f}\right)_P(\alpha) = C_m\left(ch\,t\right)\left(\widehat{A_{ch\,t}^m\left(D_\lambda^m f\right)}\right)_P(\alpha)$$

$$= C_m\left(ch\,t\right)\left(\widehat{D_\lambda^m f}\right)_P(\alpha)\left(Q_\alpha^\lambda\left(ch\,t\right)\right)^m$$

$$= \left(\alpha\left(\alpha+2\lambda\right)\right)^m C_m\left(ch\,t\right)\hat{f}_P(\alpha)\left(Q_\alpha^\lambda\left(ch\,t\right)\right)^m. \tag{3.18}$$

Consider the Gegenbauer transform of $D_\lambda^m\left(P_\nu f\right)$. Using (2.6) and (3.18) we write

$$\left(\widehat{D_\lambda^m P_\nu f}\right)_P(\alpha) = \left(\alpha\left(\alpha+2\lambda\right)\right)^m \left(\widehat{P_\nu f}\right)_P(\alpha)$$

$$= \frac{\left(\alpha\left(\alpha+2\lambda\right)\right)^m C_m\left(ch\,t\right)\widehat{P_\nu f}_P(\alpha)\left(Q_\alpha^\lambda\left(ch\,t\right)\right)^m}{C_m\left(ch\,t\right)\left(Q_\alpha^\lambda\left(ch\,t\right)\right)^m} = \frac{\left(\widehat{\varDelta_{ch\,t}^m P_\nu f}\right)_P(\alpha)}{C_m\left(ch\,t\right)\left(Q_\alpha^\lambda\left(ch\,t\right)\right)^m}. \tag{3.19}$$

Using the equality (3.19) inverse Gegenbauer transform and taking into account we have

$$D_\lambda^m P_\nu f\left(ch\,x\right) = \frac{\varDelta_{ch\,t}^m P_\nu f\left(ch\,x\right)}{C_m\left(ch\,t\right)\left(Q_\alpha^\lambda\left(ch\,t\right)\right)^m}. \tag{3.20}$$

From (1.3) it follows that the function $Q_\alpha^\lambda\left(ch\,t\right)$ is increasing. Moreover

$$\lim_{t\to\infty} Q_\alpha^\lambda\left(ch\,t\right) = \infty$$

and from the formula (see [9], p. 1934)

$$Q_\alpha^\lambda\left(ch\,t\right) = F\left(-\alpha,\,\alpha+2\lambda;\,\lambda+\frac{1}{2},sh^2\,\frac{t}{2}\right), \qquad 0 \le sh\,\frac{t}{2} \le 1$$

This function takes its minimal value at $t = 0$ (see [17], p. 1053). We have the following

$$\min Q_\alpha^\lambda\left(ch\,t\right) = Q_\alpha^\lambda\left(1\right) = F\left(-\alpha,\alpha+2\lambda;\lambda+\frac{1}{2};0\right) = 1. \tag{3.21}$$

On the other hand (see [14], Lemma 5)

$$c_1\left(m,\lambda\right)\left(sh\,\frac{t}{2}\right)^{2m} \leq C_m\left(ch\,t\right) \leq c_2\left(m,\lambda\right)\left(sh\,\frac{t}{2}\right)^{2m}. \tag{3.22}$$

Taking into account (3.21) and (3.22) in (3.20), we have

$$\|D_\lambda^m\left(P_\nu f\right)\|_{p,\lambda} \lesssim \left(sh\,\frac{t}{2}\right)^{-2m}\|\Delta_{ch\,t}^m P_\nu f\|_{p,\lambda}.$$

From Lemmas 3.7 and 3.1 it follows that

$$P_\nu f(chx) = f(chx) + \triangle_{ch\,\frac{1}{\nu}}^n f(chx) \tag{3.23}$$

then from (1.10) we have

$$\|P_\nu f\|_{p,\lambda} = \|f + \triangle_{ch\,\frac{1}{\nu}}^n f\|_{p,\lambda} \leq (2^n + 1)\|f\|_{p,\lambda}.$$

From this taking into account the commutativity of the operator $A_{ch\,t}$, we obtain

$$\|D_\lambda^m\left(P_\nu f\right)\|_{p,\lambda} \lesssim \left(sh\frac{t}{2}\right)^{-2m}\|P_\nu\left(\Delta_{ch\,t}^m f\right)\|_{p,\lambda} \lesssim \left(sh\frac{t}{2}\right)^{-2m}\|\Delta_{ch\,t}^m f\|_{p,\lambda},$$

then the inequality (3.16) is proved.

Now, let $f \in W_{p,\lambda}^m$. From (3.18) we have

$$(\widehat{\Delta_{ch\,t}^m\left(P_\nu f\right)})_P(\alpha) = C_m\left(ch\,t\right)\left(A_{ch\,t}^m \widehat{D_\lambda^m\left(P_\nu f\right)}\right)_P(\alpha). \tag{3.24}$$

Applying the inverse transformation F_P^{-1} to equality (3.24), we obtain

$$\Delta_{ch\,t}^m P_\nu f = C_m(ch\,t)A_{ch\,t}^m D_\lambda^m\left(P_\nu f\right)$$

$$= C_m\left(ch\,t\right)A_{ch\,t}^m D_\lambda^m\left(f + \Delta_{ch\,\frac{1}{\nu}}^m f\right)$$

$$= C_m\left(ch\,t\right)A_{ch\,t}^m\left(D_\lambda^m f + \Delta_{ch\,\frac{1}{\nu}}^m D_\lambda^m f\right),$$

from where taking into account (3.22) and (1.10), we have

$$\|\Delta_{ch\,t}^m P_\nu f\|_{p,\lambda} \lesssim \left(sh\,\frac{t}{2}\right)^{2m}\|A_{ch\,t}^m D_\lambda^m f\|_{p,\lambda} \lesssim \left(sh\,\frac{t}{2}\right)^{2m}\|D_\lambda^m f\|_{p,\lambda}, \tag{3.25}$$

then the inequality (3.17) is proved.

Thus Proposition 3.3 is proved.

Here is a useful corollary that follows from inequality (3.16) and the definition of the modulus of smoothness.

Corollary 3.1. *For $f \in L_{p,\lambda}$, $1 \leq p \leq \infty$, $\nu > 0$, $m \in \mathbb{N}$ the following inequality holds*

$$\|D_\lambda^m\left(P_\nu f\right)\|_{p,\lambda} \lesssim \nu^{2m}\omega_m\left(f,\frac{1}{\nu}\right)_{p,\lambda}. \tag{3.26}$$

Then it follows from the fact that

$$\frac{1}{sh^{2m}\,\frac{1}{2\nu}} \leq \frac{1}{\left(\frac{1}{2\nu}\right)^{2m}} = 4^m\,\nu^{2m}.$$

The following proposition and corollary can be regarded as analogous of the classical Boas inequality (see [41], p. 266).

Proposition 3.4. Let $f \in W^m_{p,\lambda}$, $1 \le p \le \infty$, $\nu > 0$. For every numbers δ and t such that $0 < \delta < t < \dfrac{1}{\nu}$ the following inequality

$$\left(sh\, \frac{\delta}{2} \right)^{-2m} \| \Delta^m_{ch\,\delta} f \|_{p,\lambda} \lesssim \left(sh\, \frac{t}{2} \right)^{-2m} \| \Delta^m_{ch\,t} f \|_{p,\lambda} \tag{3.27}$$

is valid.
Proof. From (3.2) we have:

$$\Delta^m_{ch\,\delta} f (ch\,x) = C_m (ch\,\delta)\, A^m_{ch\,\delta} D^m_\lambda f (ch\,x) \tag{3.28}$$

and

$$\Delta^m_{ch\,t} f (ch\,x) = C_m (ch\,t)\, A^m_{ch\,t} D^m_\lambda f (ch\,x). \tag{3.29}$$

Then the following equalities follow:

$$\| \Delta^m_{ch\,\delta} f \|_{p,\lambda} = C_m (ch\,\delta)\, \| A^m_{ch\,\delta} D^m_\lambda f \|_{p,\lambda},$$

$$\| \Delta^m_{ch\,t} f \|_{p,\lambda} = C_m (ch\,t)\, \| A^m_{ch\,t} D^m_\lambda f \|_{p,\lambda}.$$

From the continuity of the operator $A_{ch\,t}$ in $W^m_{p,\lambda}$, we have

$$\lim_{t \to 0} \| \Delta^m_{ch\,t} f / C_m (ch\,t) \|_{p,\lambda} = \lim_{\delta \to 0} \| \Delta^m_{ch\,\delta} f / C_m (ch\,\delta) \|_{p,\lambda} = \| D^m_\lambda f \|_{p,\lambda}.$$

Since $t \to 0$ involve $\delta \to 0$ that

$$\| \Delta^m_{ch\,t} f \|_{p,\lambda} / C_m (ch\,t) \sim \| \Delta^m_{ch\,\delta} f \|_{p,\lambda} / C_m (ch\,\delta) \quad \text{ïðè}\quad t \to 0, \delta \to 0$$

from this it follows that $0 < \delta < t < \dfrac{1}{\nu}$

$$C_m (ch\,t)\, \| \Delta^m_{ch\,\delta} f \|_{p,\lambda} \sim C_m (ch\,\delta)\, \| \Delta^m_{ch\,t} f \|_{p,\lambda}.$$

Taking into account (3.22), we obtain (3.27).
Thus Proposition 3.4 is proved.
Corollary 3.2. Let $f \in W^m_{p,\lambda}$, $1 \le p \le \infty$, $\nu > 0$, $m \in \mathbb{N}$. For any $t \in \left(0, \frac{1}{\nu} \right]$ inequality

$$\| D^m_\lambda f \|_{p,\lambda} \lesssim \left(sh\, \frac{t}{2} \right)^{-2m} \| \Delta^m_{ch\,t} f \|_{p,\lambda} \tag{3.30}$$

is valid. Taking into account of the continuity of operator $A_{ch\,t}$ in $W^m_{p,\lambda}$ and also relation (see [14], Lemma 5)

$$C_m (ch\,t) \sim c (m,\lambda) (ch\,t - 1)^m \text{ at } t \to 0,$$

where

$$c (m,\lambda) = \frac{1}{m!\, (2\lambda + 1) (2\lambda + 3) .... (2\lambda + 2m - 1)}.$$

From (3.28) we have

$$\lim_{\delta \to 0} \left\| \frac{\Delta^m_{ch\,\delta} f}{C_m (ch\,\delta)} \right\|_{p,\lambda} = \lim_{\delta \to 0} \| \Delta^m_{ch\,\delta} (D^m_\lambda f) \|_{p,\lambda} = \| D^m_\lambda f \|_{p,\lambda}.$$

If we take limit as $\delta \to 0$ in (3.27), then we obtain (3.30).
Inequality (3.30) is an analogue of the classical Nikol'skii-Stechkin inequality (see [29], [41]).
Taking into account that $sh\, \dfrac{t}{2} \ge \dfrac{t}{2}$ from (3.30) we obtain

$$\| D^m_\lambda f \|_{p,\lambda} \lesssim \left(\frac{2}{t} \right)^{2m} 2^m \| f \|_{p,\lambda}.$$

Putting here $t = \dfrac{1}{\nu}$, we have the inequality

$$\|D_\lambda^m f\|_{p,\lambda} \lesssim \nu^{2m} \|f\|_{p,\lambda}. \tag{3.31}$$

From this at $m = 1$ the assertion of Theorem 1.3 follows.

Proposition 3.5. Let $f \in W_{p,\lambda}^m$, $1 \le p \le \infty$, and $\nu > 0$. For any number δ and t such that $0 < \delta < t < \dfrac{1}{\nu}$ the following inequality holds:

$$\left(sh\ \frac{\delta}{2}\right)^{2m} \|\Delta_{ch\,\delta}^m P_\nu f\|_{p,\lambda} \lesssim \left(sh\ \frac{t}{2}\right)^{-2m} \|\Delta_{ch\,\delta}^m P_\nu f\|_{p,\lambda}. \tag{3.32}$$

Proof. Changing into (3.28) f by $P_\nu(f)$ we have

$$\Delta_{ch\,\delta}^m P_\nu f\,(ch\,x) = C_m\,(ch\,\delta)\,A_{ch\,\delta}^m D_\lambda^m P_\nu f\,(ch\,x).$$

From here taking into account the inequalities (3.23) and (3.30), we obtain

$$\|\Delta_{ch\,\delta}^m P_\nu f\|_{p,\lambda} \le C_m\,(ch\,\delta)\,\|D_\lambda^m P_\nu f\|_{p,\lambda}$$

$$\lesssim \left(sh\ \frac{\delta}{2}\right)^{2m} \left(sh\ \frac{t}{2}\right)^{-2m} \|\Delta_{ch\,t}^m P_\nu f\|_{p,\lambda},$$

from here (3.32) follows.

Thus Proposition 3.5 is proved.

Lemma 3.12. *Let $g \in W_{p,\lambda}^m$, $1 \le p \le \infty$, $t > 0$. Then the following estimate holds:*

$$\omega_m\,(g,t)_{p,\lambda} \lesssim \left(sh\ \frac{t}{2}\right)^{2m} \|D_\lambda^m g\|_{p,\lambda}. \tag{3.33}$$

Proof. Let $u \in (0, t]$, $\nu = \dfrac{1}{t}$. From equality

$$\Delta_{ch\,u}^m g = \Delta_{ch\,u}^m\,(g - P_\nu f) + \Delta_{ch\,u}^m P_\nu f,$$

using consecutive the inequalities (1.10), (3.17) and Theorem 3.1 we obtain

$$\|\Delta_{ch\,u}^m g\|_{p,\lambda} \le 2^m \|g - P_\nu f\|_{p,\lambda} + \|\Delta_{ch\,u}^m P_\nu f\|_{p,\lambda}$$

$$\le 2^m E_\nu\,(f)_{p,\lambda} + \left(sh\ \frac{u}{2}\right)^{2m} \|D_\lambda^m g\|_{p,\lambda} \lesssim t^{2m} \|D_\lambda^m g\|_{p,\lambda}$$

$$+ \left(sh\ \frac{t}{2}\right)^{2m} \|D_\lambda^m g\|_{p,\lambda} \lesssim \left(sh\ \frac{t}{2}\right)^{2m} \|D_\lambda^m g\|_{p,\lambda},$$

from this the inequality (3.33) follows .

Thus Lemma 3.12 is proved.

Proposition 3.6. For any function $f \in L_{p,\lambda}$, $1 \le p \le \infty$ and number $m \in \mathbb{N}$, $t > 0$ the inequality is valid:

$$\omega_m\,(f,t)_{p,\lambda} \lesssim K_m\,(f,t). \tag{3.34}$$

Proof. Let $g \in W_{p,\lambda}^m$. Using smoothness property of the modulus and Lemma 3.12, we obtain

$$\omega_m\,(f,t)_{p,\lambda} \le \omega_m\,(f - g,\,t)_{p,\lambda} + \omega_m\,(g,t)_{p,\lambda}$$

$$\lesssim \|f - g\|_{p,\lambda} + \left(sh\ \frac{t}{2}\right)^{2m} \|D_\lambda^m g\|_{p,\lambda} \lesssim \|f - g\|_{p,\lambda} + t^{2m} \|D_\lambda^m g\|_{p,\lambda},$$

as $sh\ t \sim t$ at $t \to 0$. Taking the infimum over all $g \in W_{p,\lambda}^m$, we obtain inequality (3.34).

Thus Proposition 3.6 is proved.

4   Proof of the Theorems 1.1 and 1.2.

We now prove an analogue of Jackson's direct theorem, from which we shall deduce Theorem 1.1.

Theorem 4.1. *For $f \in L_{p,\lambda}$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$, $\nu > 0$ the following inequality holds*

$$E_\nu (f)_{p,\lambda} \leq \omega_k \left(f, \frac{1}{\nu} \right)_{p,\lambda} . \tag{4.1}$$

Proof. Since

$$E_\nu (f)_{p,\lambda} \leq \|f - P_\nu f\|_{p,\lambda} , \quad \left\| \Delta_{ch \, \frac{1}{\nu}}^k f \right\|_{p,\lambda} \leq \omega_k \left(f, \frac{1}{\nu} \right)_{p,\lambda} ,$$

then it is enough to prove the inequality

$$\|f - P_\nu f\|_{p,\lambda} \leq \left\| \Delta_{ch \, \frac{1}{\nu}}^k f \right\|_{p,\lambda} . \tag{4.2}$$

for prove (4.1). From (3.23) we have

$$P_\nu f (ch \, x) = \Delta_{ch \, \frac{1}{\nu}}^n f (ch \, x) + f (ch \, x) ,$$

from this (4.2) follows. Thus Theorem 4.1 is proved.

Proof of Theorem 1.1. Let $f \in W_{p,\lambda}^m$. Taking in Theorem 4.1 $k+m$ instead of k, we obtain the following inequality

$$E_\nu (f)_{p,\lambda} \leq \omega_{k+m} \left(f, \frac{1}{\nu} \right)_{p,\lambda} . \tag{4.3}$$

From the smoothness property of the modulus it follows that

$$\omega_{k+m} (f, t)_{p,\lambda} \lesssim \nu^{-2m} \omega_k \left(D_\lambda^m f, \frac{1}{\nu} \right)_{p,\lambda} ,$$

from where we have

$$E_\nu (f)_{p,\lambda} \lesssim \nu^{-2m} \omega_k \left(D_\lambda^m f, \frac{1}{\nu} \right)_{p,\lambda} .$$

Thus the proof of the theorem is completed.

Proof of Theorem 1.2. We ought to justice of the correlation

$$K_m (f, t) \simeq \omega_m (f, t)_{p,\lambda} , \quad t \to 0. \tag{4.4}$$

If we take $g = P_\nu f$, $\nu > 0$, then from the definition of $K_m (f, t)$ it follows that

$$K_m (f, t) \leq \|f - P_\nu f\|_{p,\lambda} + t^{2m} \|D_\lambda^m (P_\nu f)\|_{p,\lambda} . \tag{4.5}$$

Using inequality (3.26) in (4.5), we obtain

$$K_m (f, t) \lesssim E_\nu (f)_{p,\lambda} + \nu^{2m} \omega_m \left(f, \frac{1}{\nu} \right)_{p,\lambda} \lesssim \left(1 + (t\nu)^{2m} \right) \omega_m \left(f, \frac{1}{\nu} \right)_{p,\lambda} . \tag{4.6}$$

Putting in (4.6) $\nu = \frac{1}{t}$, we obtain

$$K_m (f, t) \lesssim \omega_m (f, t)_{p,\lambda} . \tag{4.7}$$

The correlation (4.4) follows from (3.34) and (4.7).

Theorem 1.2 is proved.

In the following proposition which will be derived from Theorem 1.2, we state for $\omega_m (f, t)_{p,\lambda}$.

Proposition 4.1. For $f \in L_{p,\lambda}$, $t > 0$, $\delta > 0$ the relation

$$\omega_m (f, \delta t)_{p,\lambda} \lesssim \max \left\{ 1, \delta^{2m} \right\} \omega_m (f, t)_{p,\lambda} \tag{4.8}$$

is valid.

Proof. Let $0 < \delta \leq 1$, then the definition of functional we have

$$K_m (f, \delta t) = \inf \left\{ \|f - g\|_{p,\lambda} + (\delta t)^m \|D_\lambda^m f\|_{p,\lambda} : g \in W_{p,\lambda}^m \right\}$$

$$\leq \inf \left\{ \|f - g\|_{p,\lambda} + t^m \|D_\lambda^m f\|_{p,\lambda} : g \in W_{p,\lambda}^m \right\}.$$

And at $\delta > 1$

$$K_m (f, \delta t) \leq \inf \left\{ \delta^{2m} \|f - g\|_{p,\lambda} + \delta^{2m} t^m \|D_\lambda^m f\|_{p,\lambda} : g \in W_{p,\lambda}^m \right\}$$

$$= \delta^{2m} \inf \left\{ \|f - g\|_{p,\lambda} + t^m \|D_\lambda^m f\|_{p,\lambda} : g \in W_{p,\lambda}^m \right\} = \delta^{2m} K_m (f, t).$$

From this and (4.4), (4.8) follows.

5   Nikol'skii-Besov Space and Their Approximation Characteristics. The Embeddings Theorems

In this section we deal with the spaces $H_{p,\lambda}^r$ and $B_{p,q,\lambda}^r$ of Nikol'skii-Besov type and describe them in terms of the best approximation by functions on class $\mathfrak{M}(\nu, p, \lambda)$. The definition of spaces $H_{p,\lambda}^r$ and $B_{p,q,\lambda}^r$ can be found in Section 1. We will show that these spaces are Banach spaces. For these at first we will prove the completeness of the space $L_{p,\lambda}$.

Proposition 5.1. The space $L_{p,\lambda}$, $1 \leq p < \infty$ is complete.

Proof. Let $\{f_n(chx)\}$, $n \in N$ be the sequence of functions in the space $L_{p,\lambda}[0, \infty]$ such that

$$\|f_n - f_m\|_{p,\lambda} \to 0 \quad \text{as} \ n, m \to \infty.$$

From this it follows that

$$\int_0^\infty |f_n(chx) - f_m(chx)|^p \, sh^{2\lambda} x \, dx \to 0 \quad \text{as} \ n, m \to \infty.$$

From the sequence $\{f_n(chx)\}$ we choose a subsequence $\{f_{n_k}(chx)\}$ such that

$$A = \sum_{k=1}^\infty \int_0^\infty \left| f_{n_{k+1}}(chx) - f_{n_k}(chx) \right|^p \, sh^{2\lambda} x \, dx < \infty.$$

Then, for any $\ell \in N$, we can write

$$\int_0^\infty \left(\sum_{k=1}^\ell \left| f_{n_{k+1}}(chx) - f_{n_k}(chx) \right| \right)^p sh^{2\lambda} x \, dx$$

$$\leq \sum_{k=1}^\ell \int_0^\infty \left| f_{n_{k+1}}(chx) - f_{n_k}(chx) \right|^p \, sh^{2\lambda} \, dx \leq A.$$

By the Lebesgue's dominated convergence theorem as $\ell \to \infty$, we obtain

$$\int_0^\infty \left(\sum_{k=1}^\infty \left| f_{n_{k+1}}(chx) - f_{n_k}(chx) \right| \right)^p sh^{2\lambda} x \, dx \leq A. \tag{5.1}$$

Therefore,

$$\sum_{k=1}^{\infty} \left| f_{n_{k+1}}(chx) - f_{n_k}(chx) \right| < \infty \quad \text{for a.e. } x \in [0, \infty).$$

Hence, in view of the completeness of real axis, the sequence $\{f_{n_k}(chx)\}$ is convergent for a.e. $x \in [0, \infty)$.

Denote by f the measurable function defined as

$$f(chx) = \begin{cases} \lim_{k \to \infty} f_{n_k}(chx) & \text{for a.e. } x \in [0, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have proved that $f_{n_k}(chx) \to f(chx)$ for a.e. $x \in [0, \infty)$. By Minkowski inequality and inequality (5.1), for all $k \in N$ we have

$$\|f_{n_k}\|_{p,\lambda} = \left\| f_{n_1} + \sum_{\nu=1}^{k-1} \left(f_{n_{\nu+1}} - f_{n_\nu} \right) \right\|_{p,\lambda} \leq \left\| |f_{n_1}| + \sum_{\nu=1}^{k-1} \left| f_{n_{\nu+1}} - f_{n_\nu} \right| \right\|_{p,\lambda}$$

$$\leq \|f_{n_1}\|_{p,\lambda} + \left\| \sum_{\nu=1}^{k-1} \left| f_{n_{\nu+1}} - f_{n_\nu} \right| \right\|_{p,\lambda} \leq \|f_{n_1}\|_{p,\lambda} + A.$$

Passing to the limit as $k \to \infty$, we obtain that

$$\|f_{n_k}\|_{p,\lambda} \leq const, \quad \text{a.e. } f \in L_{p,\lambda}[0, \infty).$$

Let us now show that

$$\|f - f_{n_k}\|_{p,\lambda} \to 0 \quad \text{as } k \to \infty.$$

Indeed taking into account (5.1) we have

$$\|f - f_{n_k}\|_{p,\lambda} = \left\| \sum_{\nu=k}^{\infty} \left(f_{n_{\nu+1}} - f_{n_\nu} \right) \right\|_{p,\lambda}$$

$$\leq \left\| \sum_{\nu=k}^{\infty} \left| f_{n_{\nu+1}} - f_{n_\nu} \right| \right\|_{p,\lambda} \leq \left\| \sum_{\nu=1}^{\infty} \left| f_{n_{\nu+1}} - f_{n_\nu} \right| \right\|_{p,\lambda} \leq A^{\frac{1}{p}}.$$

By the Lebesgue's dominated convergence theorem

$$\|f - f_{n_k}\|_{p,\lambda} \to 0 \quad \text{as } k \to \infty.$$

Finally, we prove that the sequence $\{f_{n_k}(chx)\}$ is convergent in $L_{p,\lambda}$. Since the sequence $\{f_{n_k}(chx)\}$ is fundamental, it follows that for a given $\varepsilon > 0$ there exists a number N_ε such that, for all $m, n > N_\varepsilon$

$$\|f_n - f_m\|_{p,\lambda} < \frac{\varepsilon}{2}.$$

Choose the number $n_k > N_\varepsilon$ so that

$$\|f - f_{n_k}\|_{p,\lambda} < \frac{\varepsilon}{2}.$$

But then we have

$$\|f - f_n\|_{p,\lambda} \leq \|f - f_{n_k}\|_{p,\lambda} + \|f_n - f_{n_k}\|_{p,\lambda} < \varepsilon.$$

The proof of Proposition 5.1 is complete.

Now we prove the completeness of the space $H_{p,\lambda}^r$. By inequality (3.31) for all $f_n, f_m \in H_{p,\lambda}^r$ we have

$$\|D_\lambda^s f_n - D_\lambda^s f_m\|_{p,\lambda} = \|D_\lambda^s (f_n - f_m)\|_{p,\lambda} \lesssim \|f_n - f_m\|_{p,\lambda}. \tag{5.2}$$

We will show that $H_{p,\lambda}^r$ is a Banach space with the norm

$$\|f\|_{H_{p,\lambda}^r} = \|f\|_{p,\lambda} + \sup_{\delta > 0} \frac{\omega_k \left(D_\lambda^s f, \delta \right)_{p,\lambda}}{\delta^{r-2s}}. \tag{5.3}$$

We can write

$$\|f_n - f_m\|_{H^r_{p,\lambda}} = \|f_n - f_m\|_{p,\lambda} + \sup_{\delta > 0} \frac{\omega_k \left(D^s_\lambda f_n - D^s_\lambda f_m, \delta\right)_{p,\lambda}}{\delta^{r-2s}}. \tag{5.4}$$

Since (see Proposition 3.1 property 3)

$$\omega_k \left(D^s_\lambda f, \delta\right)_{p,\lambda} \leq 2^k \|D^s_\lambda f\|_{p,\lambda}, \tag{5.5}$$

then

$$\frac{\omega_k \left(D^s_\lambda f, \delta\right)_{p,\lambda}}{\delta^{r-2s}} \lesssim \|D^s_\lambda f\|_{p,\lambda}. \tag{5.6}$$

At $\delta \geq 1$ the inequality (5.6) is obvious, but for $0 < \delta < 1$ it is submitted to condition $2^{m-1} < \frac{1}{\delta} < 2^m$ (evidently, $m \geq 1$). Taking into account (5.2) and (5.6) in (5.4), we obtain

$$\|f_n - f_m\|_{H^r_{p,\lambda}} \leq \|f_n - f_m\|_{p,\lambda} + \|D^s_\lambda f_n - D^s_\lambda f_m\|_{p,\lambda} \lesssim \|f_n - f_m\|_{p,\lambda}. \tag{5.7}$$

Let for every $\varepsilon > 0$, $\|f_n - f_m\|_{p,\lambda} < \varepsilon$, at $n, m > N$. Then from (5.7) it follows that

$$\|f_n - f_m\|_{H^r_{p,\lambda}} < \varepsilon, \ \text{for} \ n, \ m > N.$$

Because of completeness of the space $L_{p,\lambda}$ we are able to write

$$\|f_n - f\|_{H^r_{p,\lambda}} \leq \|f_n - f\|_{p,\lambda} < \varepsilon, \ \text{ïðè} \ n > N,$$

from this the completeness of the space $H^r_{p,\lambda}$ follows.

Now we show that $B^r_{p,q,\lambda}$ is a Banach space. Let for the class $B^r_{p,q,\lambda}$ for every $\varepsilon > 0$

$$\|f_n - f_m\|_{B^r_{p,q,\lambda}} = \|f_n - f_m\|_{p,\lambda} + \left(\int_0^\infty \frac{\omega_k \left(D^s_\lambda f_n - D^s_\lambda f_m, \delta\right)^q_{p,\lambda}}{\delta^{(r-2s)q}} \frac{d\delta}{\delta}\right)^{\frac{1}{q}} < \varepsilon,$$

$$(n, m > N, \ k \geq 1; \ s = 1, 2, ...).$$

By the Lebesgue's dominated convergence theorem as $m \to \infty$, we obtain

$$\|f_n - f\|_{B^r_{p,q,\lambda}} = \|f_n - f\|_{p,\lambda} + \left(\int_0^\infty \frac{\omega_k \left(D^s_\lambda f_n - D^s_\lambda f, \delta\right)_{p,\lambda}}{\delta^{(r-2s)q}} \frac{d\delta}{\delta}\right)^{\frac{1}{q}} < \varepsilon$$

for $n > N$. Then the completeness of the space $B^r_{p,q,\lambda}$ follows. For these spaces the invested theorems are obtained. Let $c_1, c_2, ...$ denote positive constants that are different in different places and can depend on some parameters.

To prove inverse theorems of approximation theory we use inequalities of Bernstein type. Moreover for the inequality (3.31) we need still one inequality of Bernstein type.

Lemma 5.1. *For $\Phi \in \mathfrak{M}(\nu, p, \lambda)$ and $t > 0$ the following inequality holds*

$$\|\Delta^m_{ch\,t} \Phi\|_{p,\lambda} \lesssim (\nu t)^{2m} \|\Phi\|_{p,\lambda}. \tag{5.8}$$

Proof. From inequalities (3.3) and (3.31) it follows that

$$\|\Delta^m_{ch\,t} \Phi\|_{p,\lambda} \lesssim t^{2m} \|D^m_\lambda \Phi\|_{p,\lambda} \lesssim (t\nu)^{2m} \|\Phi\|_{p,\lambda}. \tag{5.9}$$

Inequality (5.9) implies that (5.8) holds. Thus Lemma 5.1 is proved.

In what follows $k, s \in \mathbb{Z}_+$, $r > 0$, $2k > r - 2s$, $\nu \geq 1$, $1 \leq q \leq \infty$.
We put for brevity $\|\cdot\| := \|\cdot\|_{p,\lambda}$.

Proof of Theorem 1.4 the same now of the Theorem 1.2 in [13].

Let

$$\tilde{h}^r_{p,\lambda}(f) := \sup_{\nu \geq 1} \nu^r E_\nu(f)_{p,\lambda}.$$

It follows from Theorem 1.4 that $f \in L_{p,\lambda}$ belongs to $H^r_{p,\lambda}$ if and only if $\tilde{h}^r_{p,\lambda}(f) < \infty$, and the norm in $H^r_{p,\lambda}$ is equivalent to the norm

$$^1 \|f\|_{H^r_{p,\lambda}} := \|f\|_{p,\lambda} + \tilde{h}^r_{p,\lambda}(f).$$

In particular for different k, s are such that $2k > r - 2s > 0$, the spaces $H^r_{p,\lambda}$ coincide and their norms are equivalent.

In the following theorem we obtain various equivalent norms in the spaces $B^r_{p,q,\lambda}$. In particular some of them will follow Theorem 1.5. As before let $r > 0$, $a > 1$ be real number, k and s be arbitrary non negative integers such that $2k > r - 2s > 0$. We shall say that a function $f(x)$ belongs to the space $^jB^r_{p,q,\lambda}$, $j = 1, 2, 3, 4$, if $f \in L_{p,\lambda}$ and the seminorm $^jb^r_{p,q,\lambda}$, is finite, where

$$^1b^r_{p,q,\lambda}(f) := \begin{cases} \left(\int\limits_0^\infty \frac{\left(\omega_k(D^s_\lambda f, \delta)_{p,\lambda} \right)^q}{\delta^{(r-2s)q}} \frac{d\delta}{\delta} \right)^{\frac{1}{q}} & \text{if} \quad 1 \leq q < \infty, \\ \sup\limits_{\delta > 0} \frac{\omega_k(D^s_\lambda f, \delta)_{p,\lambda}}{\delta^{r-2s}} & \text{if} \quad q = \infty, \end{cases}$$

$$^2b^r_{p,q,\lambda}(f) := \begin{cases} \left(\int\limits_0^a \frac{\left(\omega_k(D^s_\lambda f, \delta)_{p,\lambda} \right)^q}{\delta^{(r-2s)q}} \frac{d\delta}{\delta} \right)^{\frac{1}{q}} & \text{if} \quad 1 \leq q < \infty, \\ \sup\limits_{0 < \delta \leq a} \delta^{-(r-2s)} \omega_k(D^s_\lambda f, \delta)_{p,\lambda} & \text{if} \quad q = \infty, \end{cases}$$

$$^3b^r_{p,q,\lambda}(f) := \begin{cases} \left(\sum\limits_{j=0}^\infty a^{jrq} \left(E_{a^j}(f)_{p,\lambda} \right)^q \right)^{\frac{1}{q}} & \text{if} \quad 1 \leq q < \infty, \\ \sup\limits_{j \in Z_+} a^{jr} E_{a^j}(f)_{p,\lambda} & \text{if} \quad q = \infty, \end{cases}$$

$$^4b^r_{p,q,\lambda}(f) := \begin{cases} \inf \left(\sum\limits_{j=0}^\infty a^{jrq} \|Q_{a^j}\|^q_{p,\lambda} \right)^{\frac{1}{q}} & \text{if} \quad 1 \leq q < \infty, \\ \inf \sup\limits_{j \in Z_+} a^{jr} \|Q_{a^j}\|_{p,\lambda} & \text{if} \quad q = \infty. \end{cases}$$

The infimum is taken over all representations of f in the form of series

$$f(x) = \sum_{j=0}^\infty Q_{a^j}(x), \ Q_{a^j}(x) \in \mathfrak{M}(a^j, p, \lambda)$$

convergent in $L_{p,\lambda}$. The $^jB^r_{p,q,\lambda}$ are Banach spaces with respect to the norms

$$\|f\|_{^jB^r_{p,q,\lambda}} := \|f\|_{p,\lambda} +^j b^r_{p,q,\lambda}.$$

Theorem 5.1. *The spaces $^jB^r_{p,q,\lambda}$, $j = 1, 2, 3, 4$ coincide and their norms (5.19) are equivalent (that is, the Banach spaces $^jB^r_{p,q,\lambda}$ are equivalent).*

We mark that Theorem (1.5) follows from the equivalence of the Banach spaces $^1B^r_{p,q,\lambda}$ and $^3B^r_{p,q,\lambda}$. For brevity we use the notation $^jB^r := {}^jB^r_{p,q,\lambda}$, $^jb := {}^jb^r_{p,q,\lambda}$, $E_N(f) := E_N(f)_{p,\lambda}$, $\|f\| := \|f\|_{p,\lambda}$ and so on. The expression $V_1 \hookrightarrow V_2$ means that the Banach space V_1 is embedded in the Banach space V_2.

Proof of Theorem 5.1 is analogous of the Theorem 5.1 from [13].

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