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Point Rupture Solutions of a Class of Quasi-linear Elliptic Equations

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Abstract Let Ω be a region in $\mathbb{R}^2$ and f be a positive C^1 function satisfying

$$\lim_{u\to 0^+} f(u) = \infty.$$

We consider the quasi-linear elliptic equations of the form

$$\operatorname{div}(a(u)\nabla u) = \frac{a'(u)}{2}|\nabla u|^2 + f(u)$$

where a is a positive C^1 function. Motivated by the thin film equations, a solution u is said to be a point rupture solution if for some $p \in \Omega$, $u(p) = 0$ and $u(p) > 0$ in $\Omega\setminus\{p\}$. Our main result is a sufficient condition on a and f for the existence of radial point rupture solutions.

Keywords: Thin film, point rupture solution, radial solution, singular elliptic equation, quasilinear elliptic equation.

1 Introduction

Let Ω be a region in $\mathbb{R}^2$, and f be a smooth function defined on $(0,\infty)$ satisfying

$$\lim_{s\to 0^+} f(s) = \infty, \tag{1.1}$$

we consider the quasi-linear elliptic equations of the form

$$\operatorname{div}(a(u)\nabla u) = \frac{a'(u)}{2}|\nabla u|^2 + f(u) \tag{1.2}$$

where the terms depending upon a are formally associated with the functional

$$\int_\Omega a(u)|\nabla u|^2$$

which can be viewed as a minimizing problem in presence of a Riemannian metric tensor depending upon the unknown u itself.

Motivated by the studies of thin film equations, a solution to (1.2) is said to be a point rupture solution if for some $p \in \Omega$, $u(p) = 0$ and $u(x) > 0$ for any $x \in \Omega\setminus\{p\}$. Our main result is the existence of a radial rupture solution:

Theorem 1. *Assume that for some $\sigma^* > 0$, $a \in C^1[0,\sigma^*]$, $f \in C^1(0,\sigma^*]$ are positive functions such that for some positive constants $m < M$,*

$$m \le a(u) \le M$$

holds for any $u \in [0,\sigma^*]$ and f is monotone decreasing function on $(0,\sigma^*]$ satisfying

$$\frac{u}{G(u)f(u)} \in L^1[0,\sigma^*] \tag{1.3}$$

where

$$G(u) = \int_0^u \frac{1}{f(s)}\mathrm{d}s.$$

Then there exists $r^* > 0$ *and a radial point rupture solution* u_0 *to* (1.2) *in* $B_{r^*}(0)$ *such that* $u_0 = u_0(r)$ *is continuous on* $[0, r^*]$,

$$u_0(0) = 0,\ u_0(r) > 0 \text{ for any } r \in (0, r^*].$$

Moreover, $u_0 \in H^1(B_{r^*}(0))$ *and* u_0 *is a weak solution to* (1.2) *in the sense that for any* $\varphi \in C_0^\infty(B_1(r^*))$,

$$\int_{B_1(r^*)} a(u_0)\nabla u_0 \nabla\varphi + \frac{a'(u_0)}{2}|\nabla u_0|^2\varphi + f(u_0)\varphi = 0.$$

When $a \equiv 1$, (1.2) is reduced to the simpler form

$$\Delta u = f(u)$$

and its rupture solution has been investigated in [4], [6] when $f(u) = u^{-\alpha} - 1$, $\alpha > 1$ which has application to the van der Waals force driven thin films, in [5] with f satisfying the growth condition (1.3) and in [2] when the space dimension ≥ 3. We also remark here that the uniqueness result for general functions a and f is still open. (1.2) has also been studied by F. Gladiali and M. Squassina [1] where they are interested in the so called explosive solutions.

2 Proof of the Main Result

We consider the quasi-linear equations of the form

$$\operatorname{div}(a(u)\nabla u) = \frac{a'(u)}{2}|\nabla u|^2 + f(u) \tag{2.1}$$

in a region $\Omega \subset \mathbb{R}^2$ where for some $\delta^* > 0$, $a \in C^1[0, \delta^*]$ and $f \in C^1(0, \delta^*]$ are positive functions such that for some positive constants $m < M$,

$$m \leq a(u) \leq M \text{ holds for any } u \in [0, \delta^*].$$

Let g be the unique solution to the Cauchy problem

$$g' = \frac{1}{\sqrt{a(g)}},\ g(0) = 0,$$

and let v be a solution to

$$\Delta v = h(v) \tag{2.2}$$

where

$$h(v) = \frac{f(g(v))}{\sqrt{a(g(v))}}.$$

Define

$$u = g(v).$$

We have

$$\nabla u = g'(v)\nabla v = \frac{1}{\sqrt{a(g)}}\nabla v,$$

hence

$$\nabla v = \sqrt{a(u)}\nabla u,$$

which leads to

$$\Delta v = \sqrt{a(u)}\Delta u + \frac{1}{2}\frac{1}{\sqrt{a(u)}}a'(u)|\nabla u|^2.$$

Hence (2.2) implies

$$\sqrt{a(u)}\Delta u + \frac{1}{2}\frac{1}{\sqrt{a(u)}}a'(u)|\nabla u|^2 = \frac{f(u)}{\sqrt{a(u)}}$$

which is equivalent to (2.1). Hence, (2.1) admits a point rupture solution if and only if (2.2) has a point rupture solution.

Noticing that $h(v) = \frac{f(g(v))}{\sqrt{a(g(v))}}$ is not necessary monotone decreasing in v. However, the boundedness of a and the monotone properties of f and g implies that

$$\frac{1}{\sqrt{M}} f(g(v)) \le h(v) \le \frac{1}{\sqrt{m}} f(g(v)) \text{ for any } v \in \left[0, g^{-1}(\delta^*)\right],$$

i.e., h is bounded by two monotone decreasing functions.

We have the following existence result on rupture solutions to (2.2):

Proposition 1. *Let $\sigma^* > 0$ and h_1, $h_2 \in C^1(0, \sigma^*]$ be monotone decreasing functions such that*

$$0 < h_1 \le h_2 \text{ on } (0, \sigma^*]$$

and

$$\lim_{v \to 0^+} h_1(v) = \lim_{v \to 0^+} h_2(v) = \infty.$$

Let $h \in C^1(0, \sigma^*]$ satisfy

$$h_1 \le h \le h_2 \text{ on } (0, \sigma^*].$$

Let

$$G_1(v) = \int_0^v \frac{1}{h_1(s)} \mathrm{d}s. \tag{2.3}$$

Assume in addition that

$$\frac{h_2}{h_1} \in L^1[0, \sigma^*] \text{ and } \frac{\int_0^v \frac{h_2(s)}{h_1(s)} \mathrm{d}s}{G_1(v) h_1(v)} \in L^1[0, \sigma^*]. \tag{2.4}$$

Then there exists $r^* > 0$ and a radial point rupture solution v_0 to

$$\Delta v = h(v) \tag{2.5}$$

in $B_{r^*}(0)$ such that $v_0 = v_0(r)$ is continuous on $[0, r^*]$,

$$v_0(0) = 0,\ v_0(r) > 0 \text{ for any } r \in (0, r^*].$$

Moreover, v_0 is monotone increasing and

$$G_1^{-1}\left(\frac{1}{4} r^2\right) \le v_0(r) \le \int_0^{G_1^{-1}\left(\frac{1}{4}r^2\right)} \frac{\int_0^v \frac{h_2(s)}{h_1(s)} \mathrm{d}s}{G_1(v) h_1(v)} \mathrm{d}v \text{ for any } r \in [0, r^*].$$

For any $\sigma \in (0, \sigma^*)$, we use v_σ to denote the unique solution to the initial value problem

$$\begin{cases} v_{rr} + \frac{1}{r} v_r = h(v), \\ v(0) = \sigma, v'(0) = 0. \end{cases} \tag{2.6}$$

Lemma 1. *There exists $r_\sigma > 0$ such that v_σ is defined on $[0, r_\sigma]$ with $v_\sigma(r_\sigma) = \sigma^*$. Moreover, $v'_\sigma(r) > 0$ on $(0, r_\sigma]$ and*

$$G^{-1}\left(\frac{1}{4} r^2\right) \le v_\sigma(r) \le \sigma + H\left(G_1^{-1}\left(\frac{1}{4} r^2\right)\right) \text{ on } [0, r_\sigma]. \tag{2.7}$$

where

$$H(w) = \int_0^w \frac{\int_0^v \frac{h_2(s)}{h_1(s)} \mathrm{d}s}{G_1(v) h_1(v)} \mathrm{d}v.$$

Proof. For simplicity, we suppress the σ subscript in this proof. We write

$$v_{rr}+\frac{1}{r}v_r=h\left(v\right)$$

in the form of

$$\left(rv_r\right)_r=rh\left(v\right)\geq 0,$$

so we have

$$rv_r=\int_0^r sh\left(v\left(s\right)\right)\mathrm{d}s\geq 0.$$

In particular, v is monotone increasing and v can be extended whenever $h\left(v\right)$ is defined and bounded. Hence, there exists $r_\sigma>0$ such that v_σ is defined on $[0,r_\sigma]$ with $v_\sigma\left(r_\sigma\right)=\sigma^*$. Since v is monotone increasing and h_1 is monotone decreasing, we have

$$\begin{aligned}rv_r&=\int_0^r sh\left(v\left(s\right)\right)\mathrm{d}s\geq\int_0^r sh_1\left(v\left(s\right)\right)\mathrm{d}s\\&\geq h_1\left(v\left(r\right)\right)\int_0^r s\mathrm{d}s=\frac{1}{2}r^2h_1\left(v\left(r\right)\right),\end{aligned}$$

hence,

$$\frac{v_r}{h_1\left(v\right)}\geq\frac{1}{2}r.$$

Integrating again, we have

$$G_1\left(v\left(r\right)\right)\geq G_1\left(\sigma\right)+\frac{1}{4}r^2\geq\frac{1}{4}r^2.$$

where

$$G_1\left(v\right)=\int_0^v\frac{1}{h_1\left(s\right)}\mathrm{d}s.$$

Since G_1 is continuous and strictly monotone increasing, G_1^{-1} is well defined and we have

$$v\left(r\right)\geq G_1^{-1}\left(\frac{1}{4}r^2\right).$$

On the other hand, since h_2 is monotone increasing,

$$rv_r=\int_0^r sh\left(v\left(s\right)\right)\mathrm{d}s\leq\int_0^r sh_2\left(v\left(s\right)\right)\mathrm{d}s\leq\int_0^r h_2\left(G_1^{-1}\left(\frac{1}{4}s^2\right)\right)s\mathrm{d}s.$$

Let $v=G_1^{-1}\left(\frac{1}{4}s^2\right)$, we have $G_1\left(v\right)=\frac{1}{4}s^2$, and

$$\frac{1}{h_1\left(v\right)}\mathrm{d}v=\frac{1}{2}s\mathrm{d}s.$$

Hence,

$$\int_0^r h_2\left(G_1^{-1}\left(\frac{1}{4}s^2\right)\right)s\mathrm{d}s=2\int_0^{G_1^{-1}\left(\frac{1}{4}r^2\right)}\frac{h_2\left(v\right)}{h_1\left(v\right)}\mathrm{d}v.$$

Hence,

$$v_r\leq\frac{2}{r}\int_0^{G_1^{-1}\left(\frac{1}{4}r^2\right)}\frac{h_2\left(s\right)}{h_1\left(s\right)}\mathrm{d}s$$

which yields

$$
\begin{aligned}
v(r) &\leq \sigma + \int_0^r \frac{2}{s}\left[\int_0^{G_1^{-1}\left(\frac{1}{4}s^2\right)} \frac{h_2(t)}{h_1(t)}\mathrm{d}t\right]\mathrm{d}s \\
&= \sigma + \int_0^{G_1^{-1}\left(\frac{1}{4}r^2\right)} \frac{2}{s}\left[\int_0^{w} \frac{h_2(t)}{h_1(t)}\mathrm{d}t\right]\frac{2}{sh_1(w)}dw \\
&= \sigma + \int_0^{G_1^{-1}\left(\frac{1}{4}r^2\right)} \frac{\int_0^w \frac{h_2(t)}{h_1(t)}\mathrm{d}t}{G_1(w)\,h_1(w)}dw \\
&= \sigma + H\left(G_1^{-1}\left(\frac{1}{4}r^2\right)\right)
\end{aligned}
$$

where

$$
H(w) = \int_0^w \frac{\int_0^s \frac{h_2(t)}{h_1(t)}\mathrm{d}t}{G_1(s)\,h_1(s)}\mathrm{d}s
$$

and we used substitution

$$
w = G_1^{-1}\left(\frac{1}{4}s^2\right).
$$

□

The bounds on v_σ imply:

Corollary 1. *There exists $r^* > 0$ such that for any $\sigma \in \left(0, \frac{\sigma^*}{2}\right]$,*

$$
r_\sigma \geq r^*.
$$

We can take

$$
r^* = 2\sqrt{G_1\left(H^{-1}\left(\frac{\sigma^*}{2}\right)\right)}.
$$

Proof. For any $\sigma \in \left(0, \frac{\sigma^*}{2}\right]$,

$$
\begin{aligned}
\sigma^* = v_\sigma(r_\sigma) &\leq \sigma + H\left(G_1^{-1}\left(\frac{1}{4}r_\sigma^2\right)\right) \\
&\leq \frac{\sigma^*}{2} + H\left(G_1^{-1}\left(\frac{1}{4}r_\sigma^2\right)\right).
\end{aligned}
$$

Hence,

$$
H\left(G_1^{-1}\left(\frac{1}{4}r_\sigma^2\right)\right) \geq \frac{\sigma^*}{2}.
$$

Since the function H is strictly monotone increasing, we have

$$
G_1^{-1}\left(\frac{1}{4}r_\sigma^2\right) \geq H^{-1}\left(\frac{\sigma^*}{2}\right)
$$

and since G_1 is strictly monotone increasing, we have

$$
r_\sigma \geq 2\sqrt{G_1\left(H^{-1}\left(\frac{\sigma^*}{2}\right)\right)}.
$$

□

The point rupture solution can be constructed as the limit of v_σ as $\sigma \to 0$.

Proof of Proposition 1. For any $\varepsilon > 0$, v_σ, $\sigma \in \left(0, \frac{\sigma^*}{2}\right]$ is a family of uniformly bounded classical solutions to

$$\Delta v = h(v) \text{ in } \overline{B_{r^*}(0)} \backslash B_\varepsilon(0),$$

hence by a diagonal argument, there exists a sequence $\{\sigma_k\}_{k=1}^\infty \subset \left(0, \frac{\sigma^*}{2}\right]$ satisfying $\lim_{k\to\infty} \sigma_k = 0$, such that $v_{\sigma_k} \to v_0$ locally uniformly in $\overline{B_{r^*}(0)} \backslash \{0\}$ as $k \to \infty$. Now (2.7) implies

$$G_1^{-1}\left(\frac{1}{4}r^2\right) \le v_0(r) \le H\left(G_1^{-1}\left(\frac{1}{4}r^2\right)\right) \text{ on } [0, r^*].$$

Since

$$\lim_{r\to 0} H\left(G_1^{-1}\left(\frac{1}{4}r^2\right)\right) = 0,$$

it is not difficulty to see, from the bounds of v_σ and v_0, that $v_{\sigma_k} \to v_0$ uniformly in $\overline{B_{r^*}(0)}$ as $k \to \infty$. The above bounds also imply that $v_0(0) = 0$ and $v_0(r) > 0$ for any $r \in (0, r^*]$. Standard elliptic theory implies that $v_0 \in C^{2,\alpha}(B_{r^*}(0) \setminus \{0\})$ and

$$\Delta v_0 = h(v_0) \text{ in } B_{r^*}(0) \setminus \{0\}.$$

Hence v_0 is a rupture solution. □

Remark 1. *The above limit in the proof should be independent of the choice of* $\{\sigma_k\}_{k=1}^\infty$. *Actually, we expect that* $v_\sigma \to v_0$ *uniformly on* $[0, r^*]$ *as* $\sigma \to 0$. *Unfortunately, we are unable to provide a proof here.*

Even though v_0 is continuous, its derivatives have singularity at the origin. Now we investigate the behavior of v_0 near the origin:

Lemma 2. *The rupture solution* $v_0 \in H^1_{loc}(B_{r^*}(0))$ *and* $f(v_0) \in H^1_{loc}(B_{r^*}(0))$ *and*

$$\lim_{r\to 0^+} r v_0'(r) = 0. \tag{2.8}$$

Proof. For any $r \in (0, r^*)$, we have

$$(r v_0'(r))' = r f(v_0) > 0.$$

Hence, $r v_0'(r)$ is monotone increasing in $(0, r^*)$. Since $r v_0'(r) \ge 0$ in $(0, r^*)$,

$$\beta = \lim_{r\to 0^+} r v_0'(r) \ge 0$$

is well defined. If $\beta > 0$, we have for r sufficiently small, say $r \in (0, \tilde{r}]$,

$$r v_0'(r) \ge \frac{\beta}{2}$$

hence, for any $r \in (0, \tilde{r}]$,

$$v_0(r) = v_0(\tilde{r}) - \int_r^{\tilde{r}} v_0'(r)\,\mathrm{d}r \le v_0(\tilde{r}) - \int_r^{\tilde{r}} \frac{\beta}{2r}\mathrm{d}r.$$

which contradicts to the fact that v_0 is continuous at 0 if we let $r \to 0^+$. Hence $\beta = 0$ and (2.8) holds.

Next, for any $\varepsilon \in (0, r^*/2)$,

$$\begin{aligned}
&\int_{B_{r^*/2}(0)\backslash\overline{B_\varepsilon(0)}} h(v_0)\,dx \\
&= \int_{B_{r^*/2}(0)\backslash\overline{B_\varepsilon(0)}} \Delta v_0\,dx \\
&= \int_{\partial B_{r^*/2}(0)} \frac{\partial v_0}{\partial r}\mathrm{d}s_x - \int_{\partial B_\varepsilon(0)} \frac{\partial v_0}{\partial r}\mathrm{d}s_x.
\end{aligned}$$

Since

$$\lim_{\varepsilon\to 0^+}\left|\int_{\partial B_\varepsilon(0)}\frac{\partial v_0}{\partial r}\mathrm{d}s_x\right|=\lim_{\varepsilon\to 0}2\pi\varepsilon v_0'(\varepsilon)=0,$$

we have

$$\lim_{\varepsilon\to 0^+}\int_{B_{r^*/2}(0)\setminus\overline{B_\varepsilon(0)}}h(v_0)\,dx=\int_{\partial B_{r^*/2}(0)}\frac{\partial v_0}{\partial r}\mathrm{d}s_x$$

hence, $h(v_0)\in L^1_{loc}(B_{r^*}(0))$. Similarly, for any $\varepsilon\in(0,r^*/2)$,

$$\begin{aligned}&\int_{B_{r^*/2}(0)\setminus\overline{B_\varepsilon(0)}}|\nabla v_0|^2\,dx\\&=-\int_{B_{r^*/2}(0)\setminus\overline{B_\varepsilon(0)}}v_0\Delta v_0dx+\int_{\partial B_{r^*/2}(0)}v_0v_0'\mathrm{d}s_x-\int_{\partial B_\varepsilon(0)}v_0v_0'\mathrm{d}s_x\\&=-\int_{B_{r^*/2}(0)\setminus\overline{B_\varepsilon(0)}}v_0h(v_0)\,dx+\int_{\partial B_{r^*/2}(0)}v_0v_0'\mathrm{d}s_x-\int_{\partial B_\varepsilon(0)}v_0v_0'\mathrm{d}s_x,\end{aligned}$$

Letting $\varepsilon\to 0$, we have

$$\lim_{\varepsilon\to 0^+}\int_{B_{r^*/2}(0)\setminus\overline{B_\varepsilon(0)}}|\nabla v_0|^2\,dx=-\int_{B_{r^*/2}(0)}v_0h(v_0)\,dx+\int_{\partial B_{r^*/2}(0)}v_0v_0'\mathrm{d}s_x,$$

hence $|\nabla v_0|^2\in L^1_{loc}(B_{r^*}(0))$ and $v_0\in H^1_{loc}(B_{r^*}(0))$. □

Now we are ready to prove our main theorem:

Proof of Theorem 1. Let $\sigma^*=g^{-1}(\delta^*)$, and for any $v\in(0,\sigma^*]$, define

$$h_1(v)=\frac{1}{\sqrt{M}}f(g(v))\ \text{ and }h_2(v)=\frac{1}{\sqrt{m}}f(g(v)).$$

We have

$$h_1(v)\le h(v)=\frac{f(g(v))}{\sqrt{a(g(v))}}\le h_2(v)$$

on $(0,\sigma^*]$. It is easy to verify that the assumption on 1 holds for h. In particular, we have

$$\frac{h_2}{h_1}=\frac{\sqrt{M}}{\sqrt{m}}\in L^1[0,\sigma^*],$$

and for any $v\in(0,\sigma^*]$,

$$\begin{aligned}G_1(v)&=\int_0^v\frac{1}{h_1(s)}\mathrm{d}s.=\sqrt{M}\int_0^v\frac{1}{f(g(s))}\mathrm{d}s\\&=\sqrt{M}\int_0^v\frac{\sqrt{a(g)}}{f(g(s))}g'(s)\,\mathrm{d}s\\&=\sqrt{M}\int_0^{g(v)}\frac{\sqrt{a(u)}}{f(u)}du\\&\ge\sqrt{mM}\int_0^{g(v)}\frac{1}{f(u)}du=\sqrt{mM}G(g(v)),\end{aligned}$$

and

$$\begin{aligned}\frac{\int_0^v\frac{h_2(s)}{h_1(s)}\mathrm{d}s}{G_1(v)\,h_1(v)}&=\frac{\sqrt{M}}{\sqrt{m}}\frac{u}{G_1(v)\frac{1}{\sqrt{M}}f(g(v))}\\&\le\frac{\sqrt{M}}{m}\frac{v}{G(g(v))\,f(g(v))}\\&\le\frac{M}{m}\frac{g(v)}{G(g(v))\,f(g(v))}\end{aligned}$$

where we used

$$g(v) \geq \frac{1}{\sqrt{M}} v.$$

Hence,

$$\begin{aligned}
&\int_0^{\sigma^*} \frac{\int_0^v \frac{h_2(s)}{h_1(s)} \mathrm{d}s}{G_1(v) h_1(v)} \mathrm{d}v \\
&\leq \int_0^{\sigma^*} \frac{M}{m} \frac{g(v)}{G(g(v)) f(g(v))} \mathrm{d}v \\
&= \int_0^{\sigma^*} \frac{M}{m} \frac{g(v)}{G(g(v)) f(g(v))} \sqrt{a(g)} g'(v) \, \mathrm{d}v \\
&\leq \frac{M\sqrt{M}}{m} \int_0^{\delta^*} \frac{u}{G(u) f(u)} du.
\end{aligned}$$

So the growth condition in (1.3) implies that,

$$\frac{\int_0^u \frac{h_2(v)}{h_1(v)} \mathrm{d}v}{G_1(u) h_1(u)} \in L^1[0, \sigma^*].$$

Proposition 1 implies the existence of a rupture solution v_0 to (2.5), hence

$$u_0 = g(v_0)$$

is a rupture solution to (1.2). The properties for v_0 imply that $u_0 \in H^1_{loc}(B_1(r^*))$, $f(u_0) \in L^1_{loc}(B_1(r^*))$ and

$$\lim_{r \to 0^+} r u_0'(r) = 0.$$

For any any $\varphi \in C_c^\infty(B_{r^*}(0))$, we have

$$\begin{aligned}
&\int_{B_{r^*}(0)} a(u_0) \nabla u_0 \nabla \varphi dx = \lim_{\varepsilon \to 0^+} \int_{B_{r^*}(0) \setminus \overline{B_\varepsilon(0)}} a(u_0) \nabla u_0 \nabla \varphi dx \\
&= \lim_{\varepsilon \to 0^+} \left(-\int_{B_{r^*}(0) \setminus \overline{B_\varepsilon(0)}} \operatorname{div}(a(u_0) \nabla u_0) \varphi dx - \int_{\partial B_\varepsilon(0)} \left(a(u_0) \frac{\partial u_0}{\partial r} \varphi \right) \mathrm{d}s_x \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\int_{B_{r^*}(0) \setminus \overline{B_\varepsilon(0)}} \left(\frac{a'(u_0)}{2} |\nabla u_0|^2 + f(u_0) \right) \varphi dx - \int_{\partial B_\varepsilon(0)} \left(a(u_0) \frac{\partial u_0}{\partial r} \varphi \right) \mathrm{d}s_x \right) \\
&= \int_{B_{r^*}(0)} \left(\frac{a'(u_0)}{2} |\nabla u_0|^2 + f(u_0) \right) \varphi dx.
\end{aligned}$$

Hence u_0 is a weak solution to (1.2) in $B_{r^*}(0)$. □

We discuss several examples at the end of this section to get a better understanding of the technical assumption on the growth rate of h in (2.4).

Example 1.

$$h(v) = b(v) v^{-\alpha}$$

where $\alpha > 0$ *is a constant and* $b(v)$ *satisfies*

$$B_1 \leq b(v) \leq B_2$$

for some constants $0 < B_1 < B_2$. *If we take*

$$h_1 = B_1 v^{-\alpha} \text{ and } h_2 = B_2 v^{-\alpha},$$

we have

$$\frac{\int_0^v \frac{h_2(s)}{h_1(s)} \mathrm{d}s}{G_1(v) h_1(v)} = \frac{(1+\alpha) B_2}{B_1} \in L^1[0, 1].$$

Example 2. *For some* $0 < p < 1$,

$$h(v) = b(v)\, v^{p+1} e^{\frac{1}{v^p}}$$

and $b(u)$ *satisfies*

$$B_1 \leq b(v) \leq B_2$$

for some constants $0 < B_1 < B_2$. *If we take*

$$h_1 = B_1 v^{p+1} e^{\frac{1}{v^p}} \text{ and } h_2 = B_2 v^{p+1} e^{\frac{1}{v^p}},$$

we have

$$\frac{\int_0^v \frac{h_2(s)}{h_1(s)} \mathrm{d}s}{G_1(v)\, h_1(v)} = \frac{B_2 p}{B_1 v^p} \in L^1[0,1].$$

Example 3.

$$f(v) = \frac{1}{2}\left[\left(1 + \sin\frac{1}{v}\right) v^{-\alpha} + \left(1 - \sin\frac{1}{v}\right) v^{-\beta}\right]$$

where

$$0 < \alpha < \beta < \alpha + 1.$$

We take

$$f_1(v) = v^{-\alpha}, f_2(v) = v^{-\beta},$$

then we have for any $v \in (0,1]$,

$$h_1(v) \leq h(v) \leq h_2(v).$$

Hence,

$$\frac{\int_0^v \frac{h_2(s)}{h_1(s)} \mathrm{d}s}{G_1(v)\, h_1(v)} = \frac{\int_0^v s^{\alpha-\beta} \mathrm{d}s}{\frac{1}{1+\alpha} v} = \frac{1+\alpha}{1+\alpha-\beta} v^{\alpha-\beta} \in L^1[0,1]$$

since $\alpha - \beta > -1$. *In this example,* h *can't be expressed as a product of a bounded function and a monotone function.*

Example 4. *This example shows that our result is not optimal. Let*

$$h(v) = 2v^3 e^{\frac{2}{v}}$$

which is monotone decreasing near the origin and

$$\lim_{v \to 0^+} h(v) = \infty.$$

Taking

$$h_1(v) = h_2(v) = h(v),$$

we have

$$\begin{aligned} \frac{\int_0^v \frac{h_2(s)}{h_1(s)} \mathrm{d}s}{G_1(v)\, h_1(v)} &= \frac{v}{2v^3 e^{\frac{2}{v}} \int_0^v \frac{1}{2s^3} e^{-\frac{2}{s}} \mathrm{d}s} \\ &= \frac{e^{-\frac{2}{v}}}{v^2 \int_0^v \frac{1}{s^3} e^{-\frac{2}{s}} \mathrm{d}s} = \frac{e^{-\frac{2}{v}}}{\frac{1}{4}(v^2 + 2v)\, e^{-\frac{2}{v}}} \\ &= \frac{4}{v(2+v)} \notin L^1(0,\sigma] \end{aligned}$$

for any $\sigma > 0$. *However, let*

$$v = \frac{-1}{\ln r},$$

we have

$$v_r = \frac{1}{r \ln^2 r}, \; v_{rr} = -\frac{1}{r^2 \ln^2 r} - 2\frac{1}{r^2 \ln^3 r},$$

and so

$$u_{rr} + \frac{1}{r} u_r = -2\frac{1}{r^2 \ln^3 r} = 2v^3 e^{\frac{2}{v}} = h(v)$$

Hence $v = \frac{-1}{\ln r}$ *is a rupture solution to* $\Delta v = h(v)$ *even if the technical assumption is not satisfied.*

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