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Effect of Oblateness of the More Massive Primary on Periodic Orbits in the Restricted Three-Body Problem

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Abstract. The motion of a particle in the restricted three-body problem is explored by treating the more massive primary as an oblate spheroid with its equatorial plane coincident with the plane of motion of the primaries using a perturbation method. Initial conditions for the infinitesimal periodic orbits around the more massive primary are generated and the effect of oblateness on the perigee of these orbits is studied as well. It is observed that when oblateness coefficient is increased, the perigee of the orbit shifts towards both the primaries depending upon the increase in period and mass ratio. It is further noticed that during this transition, for certain periods, the perigee of the orbit remains unaltered with the increase in the oblateness coefficient.

Keywords: RTBP, Earth-Moon System, series solution, periodic orbit, oblateness effect.

1 Introduction

Important information about a dynamical system is provided by the study of the periodic orbits in the system. A periodic solution is regarded as a solution of the differential equation of motion that satisfies, in addition to the initial conditions, that after a lapse of one period both coordinates and velocities return to their initial values. The general aspects of periodic orbits in the restricted three- body problem (RTBP) are well known. The works of Stromgren's group at Copenhagen [1] and that of Moulton [2] and Darwin [3] on periodic orbits in RTBP are very important contributions. Broucke [4] had presented a large number of periodic orbits, classified into 10 families, in the Earth-Moon system RTBP. Szebehely [5] provided an excellent treatise on RTBP and discussed various aspects including periodic orbits.

Many of the problems facing physicists, engineers, and applied mathematicians involve difficulties as nonlinear governing equations, variable coefficients, and nonlinear boundary conditions at complex known or unknown boundaries that preclude solving them exactly. Consequently, solutions are approximated using numerical techniques, analytic techniques and combinations of both. New analytical techniques were discovered and became operational, including averaging methods, regularization, Lie series, KAM Theory, canonical operations in the extended phase space, resonance theory, perturbations in rectangular co-ordinates, asymptotic expansions and the theory of singular perturbations, search for new integrals, and new concepts of local and global stability, to mention just a few. Both general and special perturbation techniques benefitted from computer development. Algebraic manipulations in general, computerized Poisson and Fourier series, and compression and storage of ephemerides by Chebyshev polynomials might be mentioned as the most significant non numerical uses of computers assisting the development of general perturbation theories.

Generally in perturbation methods, we start with an integrable system whose solutions are known completely, and study a small perturbation of the unperturbed solution. Since the unperturbed and perturbed vector fields are closed, we might expect that the solutions will also be close. This is not always the case. In that case the unperturbed system is often structurally unstable. Arbitrarily small perturbations of such systems cause radical qualitative changes in the structure of solutions. Foremost, among the analytic techniques are the systematic methods of perturbations (asymptotic expansions) in terms of a small or a large parameter or co-ordinate. McCuskey [6] treated the basic mathematical description of the perturbation problem together with the application to some astronomical systems. The book "Introduction to perturbation methods" by Nayfeh [7] presented in a unified way the most of the perturbation techniques. The book by Guckenheimer and Holms [8] provides the analytical methods of averaging and perturbation theory for the study of periodically forced nonlinear oscillators. Various perturbation techniques for differential equations are described by Jordan and Smith [9] in their book “Nonlinear ordinary differential equations”. A detailed analysis of the stability properties of periodic orbits, in the cases that are relevant to the operations of electromagnetic tethers working in circular inclined orbits, has been carried out by Pelaez and Lara [10] by using an algorithm based on the Poincare method of continuation of periodic orbits.

In the framework of the planar circular RTBP, Huang [11] used the method of successive approximations to study some interesting orbits which provided a useful background for deriving periodic orbits for the Moon probing vehicle in the Earth–Moon–Sun system. Huang and Wade [12] derived two families of direct and retrograde periodic orbits that enclose both the Earth and the Moon. Their stability was examined by investigating the variations of the difference of two successive periods. Later Huang [13] established a series solution for periodic orbits in the RTBP which were revolving around the more massive primary. Recently Abouelmagd et al. [14] studied periodic and secular solutions in the RTBP under the effect of zonal harmonics of the more massive primary around the triangular points.

In this paper, we study the periodic orbits in the RTBP by Huang’s method by considering the more massive primary as an oblate spheroid with its equatorial plane coincident with the plane of motion. Two parameters, the mass ratio $\mu = m_2 / \left(m_1 + m_2\right)$, where m_1 and m_2 are masses of the more massive and smaller primary, respectively, and the oblateness coefficient $A_1 = \left(AE^2 - AP^2\right) / 5R^2$, where AE and AP are the equatorial and polar radii of the more massive primary, and R is the distance between the primaries, are chosen for series expansions. Initial conditions for the infinitesimal periodic orbits around the more massive primary are generated and the effect of oblateness on the perigee of the generated orbits is studied as well.

The paper is organized in 8 sections. Section 2 deals with the equations of motion with origin at the more massive primary. The equations of motion in polar co-ordinates are described in Section 3. Equations of perturbations are provided in Section 4. In Section 5, second order equations of motion with approximations are given. A solution of first order equations is presented in Section 6. Numerical applications are shown in Section 7 and the conclusions are drawn in Section 8.

2 Equations of Motion

The equations of motion of the problem under consideration in the dimensionless barycentric synodic coordinate system $\left(x_1, x_2\right)$ are (Sharma and Subba Rao [15])

$$\ddot{x}_1 - 2n\dot{x}_2 = \frac{\partial \Omega}{\partial x_1}, \quad \ddot{x}_2 + 2n\dot{x}_1 = \frac{\partial \Omega}{\partial x_2},$$

with

$$r_1 = \left(\left(x_1 - \mu\right)^2 + {x_2}^2\right)^{1/2}, \quad r_2 = \left(\left(x_1 + 1 - \mu\right)^2 + {x_2}^2\right)^{1/2},$$

and

$$n^2 = 1 + \frac{3}{2} A_1 \quad ,$$

where

$$\Omega = \frac{n^2}{2}\left(x_1^2 + x_2^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{A_1\left(1-\mu\right)}{2r_1^3}.$$

Shifting the origin to m_1, the equations of motion become

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d} t^2} - 2n\frac{\mathrm{d} x_2}{\mathrm{d} t} - n^2\left(x_1 + \mu\right) = -\left(\frac{\left(1-\mu\right)x_1}{r_1^3} + \frac{\mu\left(x_1 + 1\right)}{r_2^3} + \frac{3A_1\left(1-\mu\right)x_1}{2r_1^5}\right) \tag{2.1}$$

$$\frac{d^2 x_2}{dt^2} + 2n\frac{dx_1}{dt} - n^2 x_2 = -\left(\frac{(1-\mu)x_2}{r_1^3} + \frac{\mu x_2}{r_2^3} + \frac{3A_1(1-\mu)x_2}{2r_1^5}\right) \tag{2.2}$$

where

$$r_1 = \left(x_1^{\ 2} + x_2^{\ 2}\right)^{1/2}, \qquad r_2 = \left(\left(x_1 + 1\right)^2 + x_2^{\ 2}\right)^{1/2},$$

and

$$\Omega = \frac{n^2}{2}\left(x_1^2 + x_2^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{A_1(1-\mu)}{2r_1^3}.$$

3 Equation of Motion in Polar Coordinates

In the equations of motion, it is assumed that the mass $1-\mu$ is at the origin and the mass μ is at the point (-1, 0). Also r_1, r_2 are the distances of the third body from the masses 1-μ and μ, respectively. So r_1 is the distance of the third body from the origin. Let $r_1 = r$. We use the transformation

$$x_1 = r\cos\theta \tag{3.1}$$

$$x_2 = r\sin\theta \tag{3.2}$$

This converts the Equations (2.1) and (2.2) into the form

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - 2nr\frac{d\theta}{dt} = n^2 r - \frac{(1-\mu)}{r^2} - \frac{\mu r}{r^3} - \frac{3A_1(1-\mu)}{2r^4} + \mu\left(n^2 - \frac{1}{r_2^{\ 3}}\right)\cos\theta \tag{3.3}$$

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} + 2n\frac{dr}{dt} = -\mu\left(n^2 - \frac{1}{r_2^{\ 3}}\right)\sin\theta \tag{3.4}$$

4 Equations of Perturbations

When μ is equal to zero, all circles with centre at the origin are the periodic solutions of the problem. If μ is small, the periodic orbits deviate only slightly from the circular one. The deviation depends upon μ. Therefore, the periodic solution may be written as

$$r = r_0 + \mu r_{11}(t) + \mu^2 r_{22}(t) + \ldots\ldots.. \tag{4.1}$$

$$\theta = \lambda t + \mu\theta_1(t) + \mu^2\theta_2(t) + \ldots\ldots.. \tag{4.2}$$

where r_0 and λ are to be determined and are independent of time. Substitution of Equations (4.1) and (4.2) into Equations (3.3) and (3.4) provides different order of approximations.

The zeroth $\left(\mu^0\right)$ order approximation gives

$$(\lambda + n)^2 = \frac{1-\mu}{r_0^{\ 3}} \tag{4.3}$$

which is Kepler's third law in the problem of two bodies. The term is $n+\lambda$ instead of λ because the equations are expressed in rotating co-ordinate system. Also λ may have positive or negative value corresponding to the third body. Neglecting higher order terms in A_1, we get

$$(\lambda + 1)^2 + \frac{3}{2}A_1(\lambda + 1) = \frac{1-\mu}{r_0^{\ 3}} \tag{4.4}$$

If the terms involving third and higher orders of r_0 are neglected, the first order approximation yields

$$\frac{\mathrm{d}^2 r_{11}}{\mathrm{d}t^2} - 2(\lambda+n) r_0 \frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \begin{pmatrix} 3(\lambda+n)^2 r_{11} + 6\frac{r_{11}A_1}{r_0^5} + \frac{r_0}{2} + \frac{3A_1}{2r_0^4} + \left(n^2 - 1 - \frac{9}{8}r_0^2\right)\cos\lambda \\ +\frac{3}{2}r_0\cos 2\lambda t - \frac{15}{8}r_0^2\cos 3\lambda t \end{pmatrix} \tag{4.5}$$

$$r_0\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} + 2(\lambda+n)\frac{\mathrm{d}r_{11}}{\mathrm{d}t} = \left(1 - n^2 + \frac{3}{8}r_0^2\right)\sin\lambda t - \frac{3}{2}r_0\sin 2\lambda t - \frac{15}{8}r_0^2\sin 3\lambda t \tag{4.6}$$

Neglecting higher order terms in A_1 provides

$$\frac{\mathrm{d}^2 r_{11}}{\mathrm{d}t^2} - 2\left(\lambda + 1 + \frac{3}{4}A_1\right) r_0 \frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \begin{pmatrix} 3\left((\lambda+1)^2 + \frac{3}{2}A_1(\lambda+1)\right) r_{11} \\ +6\frac{r_{11}A_1}{r_0^5} + \frac{r_0}{2} + \frac{3A_1}{2r_0^4} + \left(\frac{3}{2}A_1 - \frac{9}{8}r_0^2\right)\cos\lambda t \\ +\frac{3}{2}r_0\cos 2\lambda t - \frac{15}{8}r_0^2\cos 3\lambda t \end{pmatrix} \tag{4.7}$$

$$r_0\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} + 2\left(\lambda + 1 + \frac{3}{4}A_1\right)\frac{\mathrm{d}r_{11}}{\mathrm{d}t} = \begin{pmatrix} \left(-\frac{3}{2}A_1 + \frac{3}{8}r_0^2\right)\sin\lambda t - \frac{3}{2}r_0\sin 2\lambda t \\ -\frac{15}{8}r_0^2\sin 3\lambda t \end{pmatrix} \tag{4.8}$$

5 Second Order Equations

With the same degree of approximation in regard to the series in r_0, the second order (μ^2) equations are

$$\frac{\mathrm{d}^2 r_{22}}{\mathrm{d}t^2} - 2(\lambda+n) r_0 \frac{\mathrm{d}\theta_2}{\mathrm{d}t} - 3r_{22}(\lambda+n)^2 - r_0\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}t}\right)^2 - 2(\lambda+n) r_{11}\frac{\mathrm{d}\theta_1}{\mathrm{d}t}$$
$$= \begin{pmatrix} -3(\lambda+n)^2\frac{r_{11}^2}{r_0} + \frac{r_{11}}{2} - \frac{9}{4}r_0 r_{11}\cos\lambda t + \frac{3}{2}r_{11}\cos 2\lambda t - \frac{15}{4}r_0 r_{11}\cos 3\lambda t \\ +6r_0 r_{11}A_1(\lambda+n)^2 + \frac{3A_1}{2r_0}(\lambda+n)^2 + 6r_0 r_{22}A_1(\lambda+n)^4 - 6r_0 r_{11}A_1(\lambda+n)^4 \\ -\left(n^2 - 1 + \frac{9}{8}r_0^2\right)\theta_1\sin\lambda t - 3r_0\theta_1\sin 2\lambda t + \frac{45}{8}r_0^2\theta_1\sin 3\lambda t \end{pmatrix} \tag{5.1}$$

$$r_0\frac{d^2\theta_2}{dt^2} + 2(\lambda+n)\frac{dr_{22}}{dt} + 2\frac{dr_{11}}{dt}\frac{d\theta_1}{dt} + r_{11}\frac{d^2\theta_1}{dt^2} = \begin{pmatrix} \frac{3}{4}r_0 r_{11}\sin\lambda t + \frac{3}{2}r_{11}\sin 2\lambda t \\ +\frac{15}{4}r_0 r_{11}\sin 3\lambda t + \left(1 - n^2 + \frac{3}{8}r_0^2\right)\theta_1\cos\lambda t \\ -3r_0\theta_1\cos 2\lambda t + \frac{45}{8}r_0^2\theta_1\cos 3\lambda t \end{pmatrix} \tag{5.2}$$

Similarly the equations of higher orders can be derived in terms of the solutions of the equations of the lower orders.

6 Solutions of First Order Equations

Differentiating equation (4.7) with respect to t and substituting equation (4.8) in the resulting expression, we get

$$\frac{\mathrm{d}^3 r_{11}}{\mathrm{d}t^3} + p^2 \frac{\mathrm{d}r_{11}}{\mathrm{d}t} = p_1 \sin\lambda t + p_2 \sin 2\lambda t + p_3 \sin 3\lambda t \tag{6.1}$$

where

$$p^2 = \left(\lambda + n\right)^2 \left(1 - \frac{6A_1}{r_0^2\left(1-\mu\right)}\right) \tag{6.2}$$

$$p_1 = \frac{3}{8} r_0^2 \left(5\lambda + 2n\right) + \left(3\lambda + 2n\right)\left(1 - n^2\right) \tag{6.3}$$

$$p_2 = -3r_0\left(2\lambda + n\right) \tag{6.4}$$

$$p_3 = \frac{15}{8} r_0^2 \left(5\lambda + 2n\right) \tag{6.5}$$

Solving Equation (6.1), we get

$$r_{11} = c_0 + c_1 \cos pt + c_2 \sin pt + k_1 \cos\lambda t + k_2 \cos 2\lambda t + k_3 \cos 3\lambda t \tag{6.6}$$

where c_0, c_1, c_2 are arbitrary constants and

$$k_1 = -\frac{p_1}{\lambda\left(p^2 - \lambda^2\right)} \tag{6.7}$$

$$k_2 = -\frac{p_2}{2\lambda\left(p^2 - 4\lambda^2\right)} \tag{6.8}$$

$$k_3 = -\frac{p_3}{3\lambda\left(p^2 - 9\lambda^2\right)} \tag{6.9}$$

Substituting (6.6) in Equation (4.8), we get

$$r_0 \frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} = \left(\begin{array}{c} 2\left(\lambda+n\right)pc_1 \sin pt - 2\left(\lambda+n\right)pc_2 \cos pt \\ +\left(2\left(\lambda+n\right)\lambda k_1 + 1 - n^2 + \frac{3}{8}r_0^2\right)\sin\lambda t \\ +\left(4\left(\lambda+n\right)\lambda k_2 - 3\frac{r_0}{2}\right)\sin 2\lambda t \\ +\left(6\left(\lambda+n\right)\lambda k_3 + \frac{15}{8}r_0^2\right)\sin 3\lambda t \end{array}\right) \tag{6.10}$$

Integrating, we get

$$r_0\theta_1 = D_1 \sin pt + D_2 \cos pt + l_1 \sin\lambda t + l_2 \sin 2\lambda t + l_3 \sin 3\lambda t + B_1 t + B_2 \tag{6.11}$$

where

$$D_1 = -2\frac{\left(\lambda+n\right)c_1}{p} \tag{6.12}$$

$$D_2 = -2\frac{\left(\lambda+n\right)c_2}{p} \tag{6.13}$$

$$l_1 = -\frac{1}{\lambda^2}\left(2\left(\lambda+n\right)\lambda k_1 + 1 - n^2 + \frac{3}{8}r_0^2\right) \tag{6.14}$$

$$l_2 = -\frac{1}{4\lambda^2}\left(4\left(\lambda+n\right)\lambda k_2 - \frac{3}{2}r_0\right) \tag{6.15}$$

$$l_3 = -\frac{1}{9\lambda^2}\left(6\left(\lambda+n\right)\lambda k_3 + \frac{15}{8}r_0^2\right) \tag{6.16}$$

where D_1, D_2 depend on c_1, c_2. B_1, B_2 are arbitrary constants.

Substituting the values of r_{11}, θ_1 in (4.7) and equating constant terms, we get

$$c_0 = \left(\begin{array}{c} -\frac{A_1}{2r_0} - \frac{r_0}{6(\lambda+1)^2}\left(1-\left(\frac{2}{r_0^2(1-\mu)}+\frac{3}{2(\lambda+1)}\right)A_1 - \frac{2B_1}{3(\lambda+1)}\right) \\ \times\left(1-\left(\frac{2}{r_0^2(1-\mu)}+\frac{3}{4(\lambda+1)}\right)A_1\right) \end{array} \right) \tag{6.17}$$

Thus neglecting the higher powers of A_1, the solution of (4.7) and (4.8) is obtained as

$$r_{11} = \left(\begin{array}{l} -\frac{A_1}{2r_0} - \frac{r_0}{6(\lambda+1)^2}\left(1-\left(\frac{2}{r_0^2(1-\mu)}+\frac{3}{2(\lambda+1)}\right)A_1 - \frac{2B_1}{3(\lambda+1)}\right) \\ \times\left(1-\left(\frac{2}{r_0^2(1-\mu)}+\frac{3}{4(\lambda+1)}\right)A_1\right) + c_1\cos pt + c_2\sin pt + k_1\cos\lambda t + k_2\cos 2\lambda t + k_3\cos 3\lambda t \end{array} \right) \tag{6.18}$$

and

$$r_0\theta_1 = D_1\sin pt + D_2\cos pt + l_1\sin\lambda t + l_2\sin 2\lambda t + l_3\sin 3\lambda t + B_1 t + B_2 \tag{6.19}$$

The values of p, k_1, k_2, k_3, l_1, l_2, l_3 are given by

$$p^2 = (\lambda+1)^2 + \frac{3}{2}A_1(\lambda+1)\left(1-\frac{4(\lambda+1)}{r_0^2(1-\mu)}\right) \tag{6.20}$$

$$k_1 = -\frac{3(5\lambda+2)}{8\lambda(2\lambda+1)}r_0^2 + \frac{A_1}{\lambda}\left(\frac{9}{16}\frac{5\lambda^2+5\lambda+1}{(2\lambda+1)^2}r_0^2 + \frac{3(3\lambda+2)}{(2\lambda+1)} - \frac{9}{4}\frac{(5\lambda+2)(\lambda+1)^2}{(2\lambda+1)^2(1-\mu)}\right) \tag{6.21}$$

$$k_2 = -\frac{3(2\lambda+1)}{2\lambda(3\lambda^2-2\lambda-1)}r_0 - \frac{9A_1}{\lambda(3\lambda^2-2\lambda-1)^2}\left(\frac{(7\lambda^2+4\lambda+1)}{8}r_0 - \frac{(2\lambda+1)(\lambda+1)^2}{r_0(1-\mu)}\right) \tag{6.22}$$

$$k_3 = \frac{5(5\lambda+2)}{8\lambda(8\lambda^2-2\lambda-1)}r_0^2 + \frac{15A_1}{\lambda(8\lambda^2-2\lambda-1)^2}\left(\frac{(3\lambda^2-9\lambda+3)}{16}r_0^2 + \frac{(5\lambda+2)(\lambda+1)^2}{4(1-\mu)}\right) \tag{6.23}$$

$$l_1 = \frac{3(10\lambda^2+12\lambda+3)}{8\lambda^2(2\lambda+1)}r_0^2 + \frac{9A_1}{\lambda^2(2\lambda+1)^2}\left(\begin{array}{l} \frac{1}{16}\left(\begin{array}{l} -10\lambda^3+122\lambda^2 \\ +147\lambda+42 \end{array}\right)r_0^2 \\ +\frac{(5\lambda^2-3\lambda-3)(\lambda+1)^2}{2(1-\mu)} \\ -\frac{3}{2}(6\lambda^2+8\lambda+3)(2\lambda+1) \end{array} \right) \tag{6.24}$$

$$l_2 = \frac{3(11\lambda^2+10\lambda+3)}{8\lambda^2(3\lambda^2-2\lambda-1)}r_0 + \frac{9A_1(\lambda+1)}{8\lambda^2(3\lambda^2-2\lambda-1)^2}\left(\begin{array}{c} (7\lambda^2+4\lambda+1)r_0 \\ -\frac{2(11\lambda^2+10\lambda+3)(\lambda+1)}{r_0(1-\mu)} \end{array} \right) \tag{6.25}$$

$$l_3 = -\frac{5(6\lambda^2+4\lambda+1)}{8\lambda^2(8\lambda^2-2\lambda-1)}r_0^2 - \frac{5A_1(\lambda+1)}{\lambda^2(8\lambda^2-2\lambda-1)^2}\left(\begin{array}{l} \frac{3}{16}\left(\begin{array}{l} 38\lambda^3+26\lambda^2 \\ -5\lambda-2 \end{array}\right)r_0^2 \\ -\frac{(29\lambda^2+\lambda-1)(\lambda+1)^2}{2(1-\mu)} \end{array} \right) \tag{6.26}$$

The periodic orbits around the more massive primary is given up to the first order terms in r_0 and μ by

$$r = r_0 + \mu r_{11} \tag{6.27}$$

and

$$\theta = \lambda t + \mu \theta_1 \tag{6.28}$$

where

$$r_{11} = -\frac{r_0}{6(\lambda+1)^2}\left(\begin{array}{c}\left(1-\left(\frac{2}{r_0^2(1-\mu)}+\frac{3}{2(\lambda+1)}\right)A_1-\frac{2B_1}{3(\lambda+1)}\right)\times \\ \left(1-\left(\frac{2}{r_0^2(1-\mu)}+\frac{3}{4(\lambda+1)}\right)A_1\right)-\frac{A_1}{2r_0} \\ +k_1\cos\lambda t+k_2\cos 2\lambda t+k3_1\cos 3\lambda t \\ \end{array}\right) \tag{6.29}$$

and

$$r_0\theta_1 = l_1\sin\lambda t + l_2\sin 2\lambda t + l_3\sin 3\lambda t \tag{6.30}$$

It follows from the solution given by Equations (6.18) and (6.19) that in general, a periodic solution can be obtained for any given value of λ only by setting c_1, c_2, B_1, B_2 equal to zero. Thus one periodic orbit is associated with one value of the period. All the foregoing relations with $A_1 = 0$ are the same as in Huang (1964), except for sign changes at some places due to the difference in the utilized co-ordinate system.

7 Numerical Applications

The Earth-Moon system is considered to obtain periodic orbits using Equations (5.28) and (5.29). In the $x_1 - x_2$ co-ordinate system with origin at the mass $(1-\mu)$, having μ=0.012149, $A_1 = 0$ and period T =0.23802754, the value of λ is obtained from the expression $T = 2\pi/\lambda$ as 26.396884. Substituting these values in Equation (4.4) we get r_0. The values of k_n, l_n are calculated using Equations (5.21) to (5.26), which are shown in Table 1. Substituting these values in Equations (5.27) to (5.30) and transforming the variables from r and θ to x_1 and x_2 in accordance with Equations (3.1) and (3.2) will give the values of x_1, x_2, $\dot{x}_1$, $\dot{x}_2$ when $t = 0$, which are denoted by x_0, y_0, $\dot{x}_0$, $\dot{y}_0$. For various values of period T, we have calculated x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ and C, which are presented in Table 2. The values are compared with Huang's results.

Similarly, the Sun-Jupiter system is considered to derive periodic orbits, with μ =0.0009539, A_1=0, and period T =0.23802754. The values of k_n, l_n are calculated, which are shown in Table 3. For various values of period, x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ and C are calculated and are presented in Table 4. Using the systems considered in Sharma and Subba Rao ([15], 1976), the values of x_0, $\dot{y}_0$ for periodic orbits around the (respective) more massive primaries are derived, and are shown in Table 5.

Table 1. Value of k_n, l_n for T = 0.23802754, λ = 26.396884.

k_n	0.0004249394	0.0001644862	0.000006901187
l_n	0.0008756117	0.0002296953	0.000008365787

Table 2. Value of $x_0, y_0, \dot{x}_0, \dot{y}_0$ $(\mu = 0.012149)$.

T	x_0	y_0	$\dot{x}_0$	$\dot{y}_0$	C
0.23802754	0.10959079	0.000000	0.000000	2.89272981	9.69689124
0.39999890	0.15239387	0.000000	0.000000	2.39363721	7.28328121
0.59999669	0.19580806	0.000000	0.000000	2.05037461	5.94990225
0.79999292	0.23271671	0.000000	0.000000	1.82777788	5.22927567
0.99999028	0.26506832	0.000000	0.000000	1.66576735	4.77580660
1.19999397	0.29394939	0.000000	0.000000	1.53981183	4.46403690
1.40001535	0.32005209	0.000000	0.000000	1.43763350	4.23681317
1.60007155	0.34385492	0.000000	0.000000	1.35226205	4.06419053

Table 3. Value of k_n, l_n for $T = 0.23802754$, λ =26.396884, $\mu = 0.0009539$

k_n	-0.0004281438	-0.0001651052	0.000006953228
l_n	0.0008822146	0.0002305597	-0.000008428873

Table 4. Value of $x_0, y_0, \dot{x}_0, \dot{y}_0$ $(\mu = 0.0009539)$.

T	x_0	y_0	$\dot{x}_0$	$\dot{y}_0$	C
0.23802754	-0.11000065	0.0000	0.0000	-2.90366433	9.74713571
0.39999890	-0.15296086	0.0000	0.0000	-2.40269691	7.31517351
0.59999666	-0.19653370	0.0000	0.0000	-2.05809668	5.97152838
0.79999290	-0.23358009	0.0000	0.0000	-1.83455031	5.24523535
0.99999026	-0.26605988	0.0000	0.0000	-1.67174119	4.78809624
1.1999940	-0.29506812	0.0000	0.0000	-1.54503453	4.47370815
1.4000154	-0.32130451	0.0000	0.0000	-1.44209402	4.24448737
1.6000716	-0.34525482	0.0000	0.0000	-1.35590940	4.07026362

Table 5. Value of x_0, $\dot{y}_0$ (various mass ratios with $T = 0.23802754$).

	μ	x_0	$\dot{y}_0$
1	0.0121490000	-0.10959079	-2.89272981
2	0.0002461294	-0.11002645	-2.90435293
3	0.0000807835	-0.11003248	-2.90451375
4	0.0000479677	-0.11003368	- 2.90454567
5	0.0000415283	-0.11003391	-2.90455193
6	0.0000250794	-0.11003451	-2.90456793
7	0.0000039400	-0.11003528	-2.90458849
8	0.0000032000	-0.11003531	-2.90458921
9	0.0000020390	-0.11003535	-2.90459034
10	0.0000010950	-0.11003539	-2.90459126
11	0.0000002000	-0.11003542	-2.90459213
12	0.0000001480	-0.11003542	-2.90459218
13	0.0000000659	-0.11003542	-2.90459226
14	0.0000000520	-0.11003542	-2.90459227

In order to study the effect of mass ratio on the periodic orbits, initial conditions are derived for periodic orbits around the more massive primary with mass ratios varying from 0.01 to 0.4. It is observed that for a given period, the value of x_0 decreases as mass ratio increases (Table 6). This shows that the perigee of the orbit moves towards the more massive primary as the mass ratio increases.

Considering different mass ratios, if the oblateness coefficient is increased, the value of x_0 decreases for a small period. When period is also increased, value of x_0 decreases for small mass ratios with the increase in the oblateness effect. For mass ratios from 0.015 onwards, the value of x_0 increases due to the increase in the oblateness. Values of the oblateness coefficient considered are $A_1 = 0.0$, $A_{11} = 0.000001$, $A_{12} = 0.000005$, $A_{13} = 0.00001$, $A_{14} = 0.00005$, $A_{15} = 0.0001$, $A_{16} = 0.0005$, $A_{17} = 0.001$, $A_{18} = 0.005$, A_{19}=0.01. As an example, we have considered two cases with $T = 0.23802754$ and $T = 0.59999666$. In the first case as oblateness increases, the value of x_0 decreases for all the mass ratios considered in Table 6. But in the second case, when oblateness increases, the value of x_0 increases for mass ratios considered from 0.015 onwards (Table 7). With further increase in period and oblateness, the value of x_0 decreases. Perigee variations (in kilometers) are shown in Table 8 and Table 9. Perigee of the orbit moves towards or away from the more massive primary depending on the mass ratio, period and oblateness effect, which is shown in Figures 1 and 2.

Table 6. The value of x_0 for various oblateness coefficients ($T = 0.23802754$).

μ	A_1	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
0.01	-0.109669709	-0.109669691	-0.109669618	-0.109669527	-0.109668799	-0.10966789	-0.109660613
0.012149	-0.109590787	-0.109590766	-0.109590679	-0.109590571	-0.109589706	-0.109588624	-0.109579973
0.015	-0.109485904	-0.109485878	-0.109485773	-0.109485642	-0.109484596	-0.109483287	-0.109472820
0.02	-0.109301461	-0.109301427	-0.109301291	-0.109301121	-0.109299758	-0.109298055	-0.109284429
0.03	-0.108930641	-0.108930591	-0.108930393	-0.108930145	-0.108928160	-0.108925679	-0.108905834
0.05	-0.108181104	-0.10818102	-0.108180706	-0.108180307	-0.108177120	-0.108173136	-0.108141262
0.1	-0.106259018	-0.106258870	-0.106258277	-0.106257536	-0.106251610	-0.106244202	-0.106184933
0.2	-0.102185281	-0.102185033	-0.102184039	-0.102182797	-0.102172861	-0.102160442	-0.102061081
0.3	-0.097750073	-0.097749796	-0.097748688	-0.097747302	-0.097736221	-0.097722368	-0.097611543
0.4	-0.092863436	-0.092863246	-0.092862488	-0.092861541	-0.092853959	-0.092844482	-0.092768657

Table 7. The value of x_0 for various oblateness coefficients ($T = 0.59999666$).

μ	A_1	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
0.01	-0.195947808	-0.195947805	-0.195947795	-0.195947782	-0.195947679	-0.195947551	-0.195946521
0.012149	-0.195808055	-0.195808054	-0.195808049	-0.195808043	-0.195807994	-0.195807934	-0.195807447
0.015	-0.195622312	-0.195622313	-0.195622315	-0.195622318	-0.195622343	-0.195622374	-0.195622622
0.02	-0.195295624	-0.195295628	-0.195295644	-0.195295664	-0.195295822	-0.195296019	-0.195297598
0.03	-0.194638634	-0.194638645	-0.194638689	-0.194638744	-0.194639185	-0.194639735	-0.194644140
0.04	-0.193309916	-0.193309943	-0.193310050	-0.193310185	-0.193311259	-0.193312601	-0.193323341
0.1	-0.189898224	-0.189898301	-0.189898611	-0.189898998	-0.189902094	-0.189905964	-0.189936924
0.2	-0.182648652	-0.182648888	-0.182649835	-0.182651018	-0.1826604850	-0.182672318	-0.182766979
0.3	-0.174731535	-0.174732040	-0.174734057	-0.174736579	-0.1747567530	-0.17478197	-0.174983703

In the case of fixed mass ratio (higher values), if the period is increased to a particular stage, the value of x_0 increases with oblateness. As the period is increased further, the value of x_0 decreases due to the increase in oblateness (Table 10) and this phenomenon continues. From Figures 3 and 4, it is observed that as oblateness coefficient increases, the perigee of the orbit shifts towards both the primaries. It is also observed that during this transition, for certain periods, the perigee of the orbit remains unaltered with the increase in oblateness coefficient (Table 11). Table 12 shows that difference in y_0 increases as oblateness increases and for a fixed oblateness coefficient; this difference decreases as period increases to a particular value. On further increase in period, this difference increases. For this study, the period considered are $T_1 = 0.23802754$, $T_2 = 0.039999890$, $T_3 = 0.59999666$, $T_4 = 0.79999290$, $T_5 = 0.99999026$, T_6 =1.1999940, T_7 =1.4000154 and $T_8 = 1.6000716$. For convenience, symbols $T_{19}: T = 0.68419$, $T_{20}: T = 0.68420$, $T_{201}: T = 0.684201$ etc. are used here.

Table 8. Perigee variations for $T = 0.23802754$ (each deviation multiplied by 3840000)

μ	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{17}	A_{18}	A_{19}
0.01	0	0	0	0	0	0	0	0	0
0.012149	0.013056	0.065664	0.131712	0.659712	1.319808	6.599808	13.200768	66.018048	132.07104
0.015	0.030336	0.152832	0.305664	1.531008	3.062400	15.313152	30.627840	153.172992	306.425856
0.02	0.061056	0.304512	0.609024	3.047424	6.094848	30.475008	60.951936	304.823424	609.806208
0.03	0.120576	0.603200	1.206144	6.032640	12.065200	60.328320	120.660096	603.429120	1207.178496
0.05	0.23616	1.180416	2.360832	11.80608	23.61216	118.065024	236.136576	1180.941312	2362.527360
0.1	0.49920	2 .495616	4.990848	24.955008	49.910016	249.556992	499.128960	2496.235008	4993.947264
0.2	0.883968	4.419840	8.839680	44.198784	88.397568	442.001664	884.036352	4421.476224	8846.192640
0.3	0.994176	4.970112	9.93984	49.700352	99.401088	497.027328	994.106880	4972.598784	9950.362752
0.4	0.657792	3.289344	6.579072	32.898816	65.79840	329.022720	658.119552	3293.516928	6594.33408

Table 9. Perigee variations for $T = 0.59999666$ (each deviation multiplied by 3840000).

μ	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{17}	A_{18}	A_{19}
0.01	0	0	0	0	0	0	0	0	0
0.012149	-0.004992	-0.025728	-0.052220	-0.260352	-0.520704	-2.604288	-5.208192	-26.02368	-52.003968
0.015	-0.011904	-0.061056	-0.122496	-0.612864	-1.226112	-6.131328	-12.262272	-61.27296	-122.445312
0.0	-0.024960	-0.125184	-0.250368	-1.252224	-2.504064	-12.52032	-25.038336	-125.111424	-250.020864
0.03	-0.051840	-0.260736	-0.521472	-2.608512	-5.217024	-26.083968	-52.164480	-260.660736	-520.917120
0.05	-0.112512	-0.564480	-1.129728	-5.649408	-11.298816	-56.491776	-112.975488	-564.553728	-1128.293376
0.1	-0.306816	-1.535232	-3.071232	-15.355776	-30.711168	-153.548544	-307.079040	-1534.652160	-3067.446528
0.2	-0.918528	-4.593408	-9.186816	-45.933696	-91.866624	-459.317760	-918.595200	-4591.35936	-9178.672128
0.3	-1.946496	-9.732864	-19.465728	-97.329024	-194.657664	-973.262592	-1946.46144	-9729.753984	-19453.120900
0.4	-3.620352	-18.102912	-36.206218	-181.03104	-362.060928	-1810.268928	-3620.449152	-18098.68262	-36188.455300

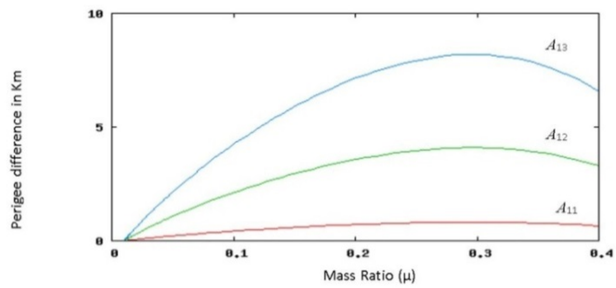

Figure 1. Perigee variations for various oblateness coefficients $A_{11} = 0.000001$, $A_{12} = 0.000005$, $A_{13} = 0.00001$ when $p = 0.23802754$ Vs mass ratio (μ).

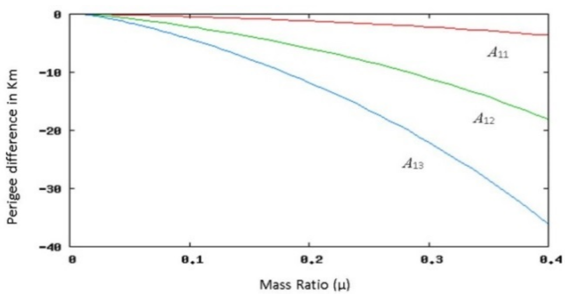

Figure 2. Perigee variations for various oblateness coefficients A_{11}=0.000001, $A_{12} = 0.000005$, $A_{13} = 0.00001$ when $p = 0.59999666$ Vs mass ratio (μ)

Table 10. Difference in x_0 with and without oblateness for various periods (μ = 0.012149, each deviation multiplied by 3840000).

A_1	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	0	0	0	0	0	0	0	0
3.7E-07	38.4	0	1.536	-1.536	-1.92	-7.68	0	0.384
5E-07	38.4	0	2.304	-1.92	-2.688	-11.52	0	0.768
0.000001	76.8	0	4.608	-3.84	-4.992	-26.88	-0.384	1.536
0.000002	192.0	38.4	9.216	-7.296	-9.6	-57.6	-0.768	2.688
0.000005	422.4	115.2	23.424	-17.664	-23.808	-142.08	-1.536	6.912
0.00001	844.8	268.8	46.464	-34.944	-47.232	-288	-3.456	13.44
0.00002	1689.6	576	93.312	-69.888	-94.08	-579.84	-6.912	26.88
0.00005	4147.2	1459.2	233.472	-174.336	-235.008	-1451.52	-17.28	67.2
0.0001	8332.8	2918.4	466.944	-348.672	-470.016	-2903.04	-34.944	134.016

Table 11. Value of x_0 from T = 0.68419 to 0.684208.

A_1	T_{19}	T_{20}	T_{201}	T_{202}	T_{203}	T_{204}
0.000000	-0.2119969670	-0.2119988323	-0.2119990211	-0.2119992098	-0.2119993986	-0.2119995762
0.00000037	-0.211996967	-0.2119988323	-0.2119990211	-0.2119992098	-0.2119993986	0.2119995762
0.00000050	-0.211996967	-0.2119988323	-0.2119990211	-0.2119992098	-0.2119993986	0.2119995762
0.00000100	-0.211996967	-0.2119988323	-0.2119990211	-0.2119992098	-0.2119993986	0.2119995762
0.00000500	-0.211996967	-0.2119988323	-0.2119990211	-0.2119992098	-0.2119993986	0.2119995762
0.00001000	-0.211996967	-0.2119988323	-0.2119990211	-0.2119992098	-0.2119993986	0.2119995762
0.00005000	-0.211996967	-0.2119988323	-0.2119990211	-0.2119992098	-0.2119993986	0.2119995762
0.00010000	-0.211996967	-0.2119988323	-0.2119990211	-0.2119992099	-0.2119993986	0.2119995763

Table 11(continued)**.** Value of x_0 from T =0.68419 to 0.684208

A_1	T_{205}	T_{2057}	T_{2059}	T_{206}	T_{207}	T_{208}
0.00000000	.2119998760	-.2119998982	-0.2119999315	-0.2119999537	-0.2120001425	0.2120003201
0.00000037	-0.2119998760	-0.2119998982	-0.2119999315	-0.2119999537	-0.2120001425	-0.2120003201
0.00000050	-0.2119998760	-0.2119998982	-.2119999315	-0.2119999537	-0.2120001425	0.2120003201
0.00000100	-0.2119998760	-0.2119998982	-0.2119999315	-0.2119999537	-0.2120001425	0.2120003201
0.00000500	-0.2119998760	-0.2119998982	-0.2119999315	-0.2119999537	-0.2120001425	0.2120003201
0.00001000	-0.2119998760	-0.2119998982	-0.2119999315	-0.2119999537	-0.2120001425	0.2120003201
0.00005000	-0.2119998760	-.2119998982	-.2119999315	-.2119999538	-0.2120001423	0.2120003202
0.00010000	-0.2119998760	-.2119998982	-.2119999316	-.2119999538	-0.2120001425	0.2120003202

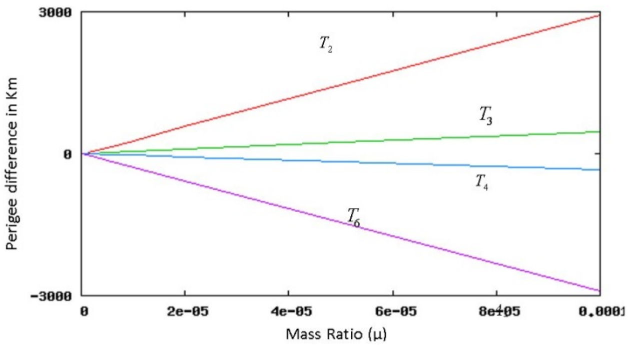

Figure 3. Perigee variations for periods $T_2 =$ 0.039999890, T_3 = 0.59999666, $T_4 =$ 0.79999290, T_6 =1.1999940 with mass ratio $\mu = 0.012149$.

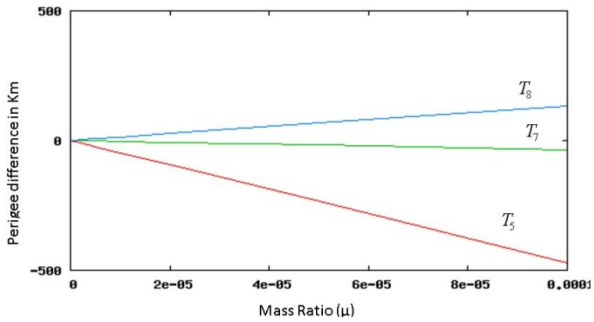

Figure 4. Perigee variations for periods $T_5 =$ 0.99999026, $T_7 =$ 1.4000154, T_8 =1.6000716 with mass ratio $\mu =$ 0.01214.

Table 12. Difference in y_0 with oblateness from that without oblateness for various periods.

A_1	T_1	T_2	T_3	A_1	T_5	T_6	T_7	T_8
0	0	0	0	0	0	0	0	0
E-07	6.345	4.042	3.102	2.773	2.585	2.538	2.538	2.538
5E-07	8.601	5.452	4.23	3.76	3.525	3.431	3.431	3.431
0.000001	17.202	10.951	8.46	7.52	7.05	6.909	6.862	6.909
0.000002	34.357	21.855	16.967	15.04	14.147	13.771	13.724	13.865
0.000005	85.869	54.708	42.441	37.6	35.344	34.451	34.31	34.639
0.00001	171.785	109.416	84.929	75.2	70.735	68.949	68.62	69.325
0.00002	343.523	218.879	169.905	150.353	141.517	137.851	137.24	138.65
0.00005	858.737	328.295	424.786	375.906	353.769	344.651	343.1	346.672
0.0001	1717.427	1094.348	849.572	751.765	707.538	689.302	686.2	693.34

8 Conclusions

In this paper, a series solution is developed for finding the initial conditions of periodic orbits by including oblateness effect of the more massive primary with the help of a perturbation technique. The results are verified with Huang's values. The effect of oblateness of the more massive primary on the initial conditions of the periodic orbits is studied by taking into consideration the variation of mass ratio (μ). As the mass ratio increases up to 0.012, the perigee of the orbit moves towards the more massive primary. With the increase in oblateness coefficient, the perigee of the orbit shifts towards both the primaries depending upon the increase in period and mass ratio. It is further noticed that during this transition, for certain periods, the perigee of the orbit remains unaltered with the increase in the oblateness coefficient. The present study will be useful in understanding the general behavior of the motion of the inner planets and asteroids as a result of the perturbation by the major planets.

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