Advances in Astrophysics, Vol. 5, No. 4, November 2020
https://dx.doi.org/10.22606/adap.2020.54003

The Radiation-Dominated Universe in Nonstandard Cosmology

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Abstract We continue the recent study of our model theory of low-density cosmology without dark matter. We assume a purely radiative spherically symmetric background and treat matter as anisotropic perturbations. Einstein's equations for the background are solved numerically. We find two irregular singular points, one is the Big Bang and the other a Big Crunch. The radiation temperature continues to decrease for another 0.21 Hubble times and then starts to increase towards infinity. Then we derive the four evolution equations for the anisotropic perturbations. In the Regge-Wheeler gauge there are three metric perturbations and a radial velocity perturbation. The solution of these equations allow a detailed discussion of the cosmic evolution of the model universe under study.

Keywords: cosmology

1 Introduction

The nonstandard cosmology discussed in the present paper is a model cosmology without dark matter or energy. To date, the existence of dark matter has not been established by the underground searches with sensitive particle detectors. On the theoretical side there are some doubts concerning the existence of dark matter in the framework of supersymmetric gauge theories due to a no-go theorem obtained in the framework of causal perturbation theory in 2008 [1]. The nontrivial rotation curves in galaxies which cannot be understood by Newtonian gravity may have an explanation within general relativity given in [2]. But there is a high price for this solution of the puzzle: one has to give up the Copernican Principle of homogeneity and instead advocate the Ptolemaic Principle that we live near the preferred place of spherical symmetry of the Universe.

Our model theory does not correspond to our actual universe, and since it basically relies on analytical methods, it does not incorporate most aspects from nucleosynthesis, baryon acoustic oscillations, galaxy cluster dynamics, galaxy formation, and many more. However, it is hoped that the model serves as a motivation to continue the search for alternatives or modificatons of the theory of general relativity.

In previous papers [3-6] the cosmic gravitational background field was described by a vacuum solution of Einstein's equations. However, for higher redshift radiation becomes important so that it is better to include the main spherically symmetric part of CMB in the background from the very beginning. This is our aim here, the paper is organized as follows. In the next section we discuss Einstein's equations for the radiation dominated background and specialize them to our Universe. In section 3 the equations are numerically integrated. They have two irregular singular points: one is the Big Bang and the other in the future is a Big Crunch. The measured Hubble diagram allows a precise calibration of the radiation dominated background. The radiation temperature continues to decrease for another 0.21 Hubble times and then it increases to infinity at the Big Crunch. In section 4 we introduce anisotropic perturbations. As in the vacuum case we use the Regge-Wheeler gauge, however the perturbed energy-momentum tensor must now have a radial velocity perturbation in addition to the density and pressure perturbations. As a consequence we have four instead of three evolution equations to be solved. In the last section these equations are derived in dimensionless form which is well suited for the numerical solution.

2 The Radiation Dominated Background

If we apply numerical calculations to our Universe the present velocity of light $c = 3 \times 10^{10}$ cm/sec plays an important role. We therefore include c in all formulas from the very beginning and write our nonstandard line element as

$$ds^2 = c^2dt^2 - X(t)^2dr^2 - R(t)^2(d\vartheta^2 + \sin^2\vartheta d\phi^2). \tag{2.1}$$

The non-zero Christoffel symbols are equal to

$$\Gamma^0_{11} = \frac{1}{c^2}\dot{X}X, \quad \Gamma^0_{22} = \frac{1}{c^2}\dot{R}R, \quad \Gamma^0_{33} = \frac{1}{c^2}\dot{R}R\sin^2\vartheta$$

$$\Gamma^1_{01} = \frac{\dot{X}}{X}, \quad \Gamma^2_{02} = \frac{\dot{R}}{R} = \Gamma^3_{03}$$

$$\Gamma^2_{33} = -\sin\vartheta\cos\vartheta, \quad \Gamma^3_{23} = \cot\vartheta \tag{2.2}$$

and we need the following components of the Riemann tensor

$$R^0_{101} = \frac{X}{c^2}\ddot{X}, \quad R^0_{202} = \frac{R}{c^2}\ddot{R}$$

$$R^1_{212} = \frac{R}{c^2}\dot{R}\frac{\dot{X}}{X} = R^3_{131}, \quad R^3_{232} = \frac{\dot{R}^2}{c^2} + 1, \tag{2.3}$$

$$R^\mu_\mu = -\frac{2}{c^2}\Big(\frac{\ddot{X}}{X} + 2\frac{\ddot{R}}{R} + 2\frac{\dot{R}\dot{X}}{RX} + \frac{\dot{R}^2}{R^2} + \frac{c^2}{R^2}\Big). \tag{2.4}$$

The mixed components of the Einstein tensor are given by

$$G^0_0 = R^0_0 - \frac{1}{2}R = \frac{2}{c^2}\frac{\dot{X}\dot{R}}{XR} + \frac{\dot{R}^2}{c^2R\check{s}} + \frac{1}{R^2} \tag{2.5}$$

$$G^1_1 = \frac{1}{c^2}\Big(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\Big) + \frac{1}{R^2} \tag{2.6}$$

$$G^2_2 = \frac{1}{c^2}\Big(\frac{\ddot{X}}{X} + \frac{\ddot{R}}{R} + \frac{\dot{X}\dot{R}}{XR}\Big) = G^3_3. \tag{2.7}$$

We want to solve Einstein's equation

$$G^\mu_\nu = \frac{8\pi G}{c^4}T^\mu_\nu \tag{2.8}$$

with the relativistic energy-momentum tensor

$$T^\mu_\nu = \mathrm{diag}\Big(\varrho, -\frac{\varrho}{3}, -\frac{\varrho}{3}, -\frac{\varrho}{3}\Big). \tag{2.9}$$

Later on $\varrho(t)$ represents the incoherent CMB radiation field while matter is treated as a perturbation.

According to (2.8) we have to solve the following three ordinary differential equations

$$2\dot{X}\dot{R}R + X(\dot{R}^2 + c^2) = \frac{8\pi G}{c^2}\varrho XR^2 \tag{2.10}$$

$$2R\ddot{R} + \dot{R}^2 + c^2 = -\frac{8\pi G}{3c^2}\varrho R^2 \tag{2.11}$$

$$R^2\ddot{X} + XR\ddot{R} + R\dot{X}\dot{R} = -\frac{8\pi G}{3c^2}\varrho R^2. \tag{2.12}$$

Here $\varrho(t)$ can be obtained from energy conservation in the form

$$\varrho = D(XR^2)^{-4/3} \tag{2.13}$$

where D is a constant of integration. Using this in (2.10-12), only two equations remain to be solved for R and X, we take the first two (2.10-11). We reduce these equations to first order equations by substituting

$$\dot{R} = u(R), \quad X(t) = v(R) \tag{2.14}$$

such that

$$\frac{d}{dt} = u\frac{d}{dR}. \tag{2.15}$$

Then we obtain

$$2u^2v'R + u^2v = v\Big(\frac{8\pi G}{c^2}R^2DR^{-8/3}v^{-4/3} - c^2\Big) \tag{2.16}$$

$$2Ru'u + u^2 = -\frac{8\pi G}{3c^2}R^2DR^{-8/3}v^{-4/3} - c^2\Big) \tag{2.17}$$

where the prime now stands for d/dR.

Our aim is to integrate the equations (2.16-17) from the present time t_0 backwards towards the Big Bang. To do so we need initial conditions u_0 and v_0 at some starting point $R(t_0) = R_0$, and the constant D too. The latter follows from the present CMB temperature $T_0 = 2.725$ K by means of the Stefan-Boltzmann law

$$\varrho_0 = a_RT_0^4 = Dv_0^{-4/3}R_0^{-8/3} \tag{2.18}$$

where

$$a_R = 7.566\times 10^{-15}\mathrm{erg}/(\mathrm{cm}^3\mathrm{K}^4)$$

is the radiation constant. This gives

$$\varrho_0 = 4.172\times 10^{-13}\mathrm{g}/(\mathrm{cm}\,\mathrm{sec}^2) \tag{2.19}$$

which allows to calculate D by (2.18). A second input is the Hubble constant defined by the redshift according to

$$-H_0 = \frac{dz}{dt}\Big|_{z=0}.$$

Since the redshift is given by

$$1+z = \frac{X(t_0)}{X(t)} = \frac{v_0}{v} \tag{2.20}$$

we have

$$H_0 = \frac{v_0}{v^2}\frac{dv}{dt}|_{z=0} = \frac{v_0}{v_0^2}v_0'\frac{dR}{dt}\Big|_{z=0} =$$

$$= \frac{u_0}{v_0}v_0' = \frac{1}{T_H} \tag{2.21}$$

where T_H is the Hubble time. The initial value v_0 follows from the radial null geodesic

$$\frac{dr}{dt} = \frac{1}{X(t)} = c(t). \tag{2.22}$$

For $t = t_0$ this yields the present light speed c so that

$$v_0 = \frac{1}{c}. \tag{2.23}$$

The derivative v_0' is obtained from the differential equation (2.16). We get the following equation for u_0

$$\frac{1}{T_H} = -\frac{u_0}{2R_0} + \frac{1}{2u_0R_0}\Big(\frac{8\pi G}{c^2}R_0^2\varrho_0 - c^2\Big). \tag{2.24}$$

As in [1] we introduce dimensionless quantities denoted by bars:

$$\bar{u} = \frac{u}{c},\quad \bar{R} = \frac{R}{cT_H} \tag{2.25}$$

such that

$$\frac{d}{dR} = \frac{1}{cT_H}\partial_{\bar{R}}.$$

Note that v is already dimensionless. Then after dividing by c^2 the differential equations (2.16-17) assume the following dimensionless form:

$$v' = \Big(D_1 \bar{R}^{-2/3} v^{-4/3} - 1\Big) v/(\bar{R}\bar{u}^2) - v/(2\bar{R}) \tag{2.26}$$

$$\bar{u}' = \Big(-\frac{D_1}{3} \bar{R}^{-2/3} v^{-4/3} - 1\Big)/(2\bar{R}\bar{u}) - \bar{u}/(2\bar{R}). \tag{2.27}$$

Now the prime means $d/d\bar{R}$ and the new constant factor is equal to

$$D_1 = 8\pi G c^{-14/3} T_H^{-2/3} D = 8\pi G \varrho_0 \frac{T_H^2}{c^2} \bar{R}_0^{8/3} \tag{2.28}$$

where (2.18) has been used. We calculate with the Hubble time [4]

$$T_H = 13.58 \times 10^9 \text{years}, \quad H_0 = 72\,\text{km}/(\text{sMpc}). \tag{2.29}$$

Then we get

$$D_1 = 1.430 \times 10^{-4} \bar{R}_0^{8/3}. \tag{2.30}$$

The small factor 10^{-4} is the reason why the radiation-dominated background does not differ much from the vacuum background. There remains R_0 to be determined, this is done in the next section.

3 Numerical Integration of Einstein's Equations

We integrate the equations (2.26-27) by using NDSolve of Mathematica. For the convenience of the reader we give the short Mathematica file:

```
z = 1
R0 = 1.49
R1 = 1.65
d1 = 1.43 * 10 ∧ (−4) * R0 ∧ (8/3)
u0 = Sqrt[R0 * R0 + d1 * R0 ∧ (−2/3) − 1] − R0
so1 = NDSolve[
{v'[x] == d1 * x ∧ (−2/3)v[x] ∧ (−4/3) − 1) * v[x]/
(2 * x * u[x] ∧ 2) − v[x] * 0.5/x,
u'[x] == (−(d1/3) * x ∧ (−2/3) * v[x] ∧ (−4/3) − 1)/
(2 * u[x] * x) − 0.5 * u[x]/x,
u[R0] == u0, v[R0] == 1},
{u, v}, {x, R0, R1}, AccuracyGoal− > 10, PrecisionGoal− > 10,
WorkingPrecision− > 15]
Plot[Evaluate[(u[x] + ((1.7111 − x) ∧ 0.5)/.so1],
{x, R0, R1}]
Plot[Evaluate[(v[x] − 2.596((1.7111 − x) ∧ 0.5)/.so1],
{x, R0, R1}]
so2 = NIntegrate[Evaluate[1/(u[x] * v[x])/.so1],
{x.R0, R1}]
5 * Log[10, −(1 + z) * %] + 43.1
```

Here we have inserted the numbers for our Universe. It is hard to imagine a more fascinating computer program. The initial value $\bar{R}_0 = 1.49$ has been chosen in such a way that for redshift $z = 1$ the correct distance modulus $\mu(z = 1) = 44.08$ in the Hubble diagram comes out. We have already used this particular measured value in our previous calculations of the Hubble diagram [2]. For completeness let us describe how the computation of the Hubble diagram goes. The redshift is given by

$$1 + z = \frac{X_{\text{obs}}}{X(t)} = \frac{v_0}{v(R)} = 1/\bar{v}(\bar{R}). \tag{3.1}$$

The radial distance is obtained from the integral

$$r(R_1) = cT_H \int_{\bar{R}_0}^{\bar{R}_1} \frac{d\bar{R}}{|\bar{u}v|}. \tag{3.2}$$

This is computed as $so2$ in the program. The luminosity distance is equal to $d_L(z) = (1+z)r(R_1)$ which then yields the distance modulus (in magnitude):

$$\mu(z) = 5\log_{10} d_L + 25. \tag{3.3}$$

Using the value $cT_H = 4164$ Mpc this gives the number 43.1 at the end of the program. To test the correctness of this calibration of the program we have computed the Hubble diagram up to $z = 10$ and compared it with the standard model value $\tilde{\mu}(z)$.The results are given in the following table. Comparing the second and third columns we find perfect agreement of the Hubble diagrams.

z	$\tilde{\mu}(z)$(mag)	$\mu(z)$(mag)	$\bar{R}$	$\bar{u}$	v
0.1	38.25	38.24	1.524	-0.350447	0.909636
0.2	39.89	39.88	1.551	-0.321348	0.834139
0.3	40.89	40.89	1.573	-0.296366	0.769321
0.4	41.62	41.62	1,5905	-0.27543	0.715005
0.5	42.20	42.20	1.6051	-0.257049	0.667317
0.6	42.69	42.68	1.6172	-0.241032	0.625763
0.7	43.10	43.10	1.6274	-0.226855	0.588984
0.8	43.46	43.46	1.6360	-0.214324	0.556476
0.9	43.79	43.78	1.6434	-0.203038	0.527196
1.0	44.08	44.08	1.6500	-0.192506	0.499873
2.0	46.05	46.04	1.6832	-0.128829	0.334697
3.0	47.22	47.21	1.6952	-0.0969382	0.251989
4.0	48.05	48.03	1.7008	-0.0779179	0.20267
5.0	48.70	48.68	1.7039	-0.0651086	0.169464
6.0	49.22	49.22	1.7059	-0.0553211	0.144097
7.0	49.67	49.64	1.707	-0.0491239	0.128038
8.0	50.05	50.01	1.7078	-0.0440776	0.114965
9.0	50.38	50.35	1.7084	-0.0398785	0.10409
10.0	50.68	50.66	1.7089	-0.0360087	0.0940699

Now comes fun. The Big Bang $z = \infty$ corresponds to a zero

$$v(R_b) = 0 \tag{3.4}$$

of $v(\bar{R})$ by (2.20). High precision calculation is necessary to find the Big Bang at $\bar{R} = R_b = 1.7111$ (we omit the bar at R_b and remember that it is dimensionless, b stands for "Bang"). However, the step-size goes to zero near $R = R_b$ which indicates a singularity and indeed, the derivatives v' and u' diverge. To understand the nature of this singularity we try an expansion in $x = R_b - \bar{R}$:

$$\bar{u} = x^n(b_0 + b_1 x + \ldots), \quad v = x^m(a_0 + a_1 x + \ldots). \tag{3.5}$$

Then in lowest order we have

$$-2\bar{u}^2 v'(R_b - x) + \bar{u}^2 v = v\Big[G_1 R_b^{-2/3}\Big(1 + \frac{2}{3}\frac{x}{R_b}\Big)v^{-4/3} - 1\Big] \tag{3.6}$$

$$-2\bar{u}\bar{u}'(R_b - x) + u^2 = -\frac{G_1}{3} R_b^{-2/3}\Big(1 + \frac{2}{3}\frac{x}{R_b}\Big)v^{-4/3} - 1 \tag{3.6}$$

where $G_1 = 8\pi G$. Substituting (3.5) and comparing the powers we obtain $n = 1/2 - 2m/3$ and the two relations

$$b_0^2 = -\frac{G_2}{2mR_b} a_0^{-4/3}, \quad b_0^2 = \frac{G_2}{6nR_b} a_0^{-4/3}$$

with $G_2 = G_1 R_b^{-2/3}$. This requires $m = 3/2$ and $n = -1/2$ but a negative

$$b_0^2 = -\frac{G_2}{3R_b} a_0^{-4/3}.$$

Consequently there is no formal power series expansion at the Big Bang, it seems to be an irregular singular point. Still we can use the numerical solution to calculate the age of the Universe:

$$T_L = \int_{\bar{R}_0}^{R_b} \frac{dr}{|u(r)|}. \tag{3.7}$$

The precise value of T_L depends on R_b, for $R_b = 1.7111$ we get $T_0 = 1.2$. Since this is in units of the Hubble time, the Universe is older in nonstandard cosmology than in the standard one.

We have tested the accuracy of the numerical solution by integrating backwards from the Big Bang to the present with the initial conditions of the forward calculation (6 digits). The most sensitive quantity is $v(\bar{R})$. For $R_1 = 1.7105$ which corresponds to a redshift $z = 19$ the accuracy is better than 1%, but for $R_1 = 1.711$ and $z = 45$ the error is allready 3%. It is a challenge to integrate until the time of last scattering at $z = 1000$.

It is a nice feature that apart from the singularities there is a simple analytic representation of the solution, this is a consequence of the small number D_1 (2.30). For $D_1 = 0$ the two equations (2.26-27) decouple and (2.27) has the solution

$$\bar{u} = -\sqrt{R_b/\bar{R} - 1} \tag{3.8}$$

where R_b now is a constant of integration. Inserting $\bar{u}$ (3.8) into (2.26) we find the linear equation

$$2\bar{R}\Big(R_b/\bar{R} - 1\Big)v' + (R_b/\bar{R})v = 0 \tag{3.9}$$

with the solution

$$v = \beta\sqrt{R_b/\bar{R} - 1} \tag{3.10}$$

where β is a second constant of integration. The value of β follows from the initial condition

$$\beta = \frac{v_0}{-u_0} = 2.596. \tag{3.11}$$

In the computer program we have compared this analytic representation with the numerical solution: For $\bar{u}(\bar{R})$ the accuracy is 10^{-5} and for $v(\bar{R})$ it is 10^{-4}, but close to R_b and $R = 0$ there are strong deviations. For $\beta = 1$ the results (3.8), (3.10) agree with our previous Schwarzschild vacuum solution, we now have a refined background.

There remains the singularity at $R = 0$ to be discussed. To understand the physics we calculate the energy density by means of the analytic representation

$$\varrho = D(vR^2)^{-4/3} = D\beta^{-4/3}(R_b - \bar{R})^{-2/3}\bar{R}^{-2}(cT_H)^{-8/3}. \tag{3.12}$$

The energy density goes to infinity at the Big Bang R_b and at $R = 0$, too. That means $R = 0$ is the so-called Big Crunch. Of course the analytic representation breaks down at the singularities, but in between it is very accurate. So we set the derivative $d\varrho/dr = 0$ and find a minimum of ϱ at

$$\bar{R}_M = \frac{3}{4}R_b = 1.283. \tag{3.13}$$

That means the CMB temperature is continuing to decrease for a Hubble distance of $(1.49 - 1.283)cT_H = 0.21cT_H$, but then it increases towards a Big Crunch where it becomes infinite. The minimal CMB energy density is 87% of the present energy density, this follows from

$$\Big(\frac{R_b - \bar{R}_M}{R_b - \bar{R}_0}\Big)^{-8/3}\Big(\frac{\bar{R}_M}{\bar{R}_0}\Big)^{-2} = 0.8684.$$

Again we test whether a formal power series at the Big Crunch is possible. It must be of the form

$$u = \frac{b_0}{R} + b_2 R + \ldots, \quad v = a_0 R + a_2 R§ + \ldots \tag{3.14}$$

where a_0 and b_0 are free integration constants. However no such expansion approximates the numerical solution because $v(R)$ goes to 0 for $R \to 0$, but the numerical solution diverges. So the Big Crunch again seems to be an irregular singular point.

Finally we must relate the metric function R to the cosmic time t. From

$$\frac{dR}{dt} = -c\sqrt{\frac{T_L}{R} - 1} \tag{3.15}$$

with

$$T_L = cT_H R_b \tag{3.16}$$

we obtain

$$ct = \int_R^{R_b} \sqrt{\frac{R}{R_b - R}} dR,$$

where the origin $t = 0$ is chosen at the Big Bang $\bar{R} = R_b$. The integral is again calculated by the substitution

$$R = T_L \sin^2 w. \tag{3.17}$$

This yields the same result as in the vacuum background case

$$t = \frac{T_L}{c}(w - \sin w \cos w - \pi). \tag{3.18}$$

Here the Big Bang corresponds to $w = \pi/2$ and $w = \pi$ is the Big Crunch, T_L (3.16) is the previous life-time of the Universe [2]. In addition we have

$$v = X = \beta\sqrt{\frac{T_L}{R} - 1} = \beta\sqrt{\frac{1}{\sin^2 w} - 1} = -\beta \cot w \tag{3.19}$$

where the sign of the square root must be chosen so that X is positive. This implies

$$u = \dot{R} = -c\sqrt{\frac{T_L}{R} - 1} = -c|\cot w| \tag{3.20}$$

and

$$\frac{dt}{dw} = \frac{2T_L}{c}\sin^2 w. \tag{3.21}$$

To relate the variable w to the redshift we remember the relation

$$1 + z = \frac{1}{X} = \frac{1}{\beta|\cot w|}. \tag{3.21}$$

4 Anisotropic Perturbations

Now we consider anisotropic perturbations of the metric

$$g_{\mu\nu} = g_{0\mu\nu} + h_{\mu\nu} \tag{4.1}$$

where $g_{0\mu\nu}$ is the radiation-dominated background metric. It is our aim to construct an approximate anisotropic solution of Einstein's equations. Again this solution must have initial conditions. As before we take these initial conditions from present astronomical observations, for example the accurately measured anisotropies of CMB, and we integrate backwards towards the Big Bang. In standard cosmology one makes assumptions about the early Universe and integrates forward to the present. Our procedure is logically better because different models can lead to almost the same observations as we have seen in the case of the Hubble diagram, and assumptions about the early Universe are always uncertain.

In the case of the vacuum background, the cosmological perturbation theory has been treated in the previous papers [2] and [3]. Now with the presence of radiation various changes are necessary, also the light speed will be included from the beginning. The first order perturbation of the Ricci tensor is given by the Palatini identity ([5], sect.9)

$$\delta R_{\mu\nu} = \frac{1}{2} g_0^{\alpha\beta} [\nabla_\nu \nabla_\mu h_{\alpha\beta} - \nabla_\alpha \nabla_\nu h_{\mu\beta} - \nabla_\alpha \nabla_\mu h_{\beta\nu} + \nabla_\alpha \nabla_\beta h_{\mu\nu}]. \tag{4.2}$$

Then the perturbation of the Einstein tensor is equal to

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \Big(\nabla^\beta \nabla^\alpha h_{\alpha\beta} - \nabla_\beta \nabla^\beta h^\alpha_\alpha - h_{\alpha\beta} R^{\alpha\beta} \Big) - \frac{1}{2} h_{\mu\nu} R. \tag{4.3}$$

Here we commute the covariant derivatives by means of the curvature tensor which gives [2]

$$\begin{aligned} 2\delta G_{\mu\nu} = -\nabla^\alpha \nabla_\alpha h_{\mu\nu} + \nabla_\nu f_\mu + \nabla_\mu f_\nu - 2R^\beta_{\mu\alpha\nu} h^\alpha_\beta - \nabla_\nu \nabla_\mu h^\alpha_\alpha - g_{\mu\nu}(\nabla^\beta f_\beta - \nabla^\beta \nabla_\beta h^\alpha_\alpha) + \\ + R^\beta_\nu h_{\mu\beta} + R^\beta_\mu h_{\nu\beta} - R h_{\mu\nu} + g_{\mu\nu} h_{\alpha\beta} R^{\alpha\beta} \end{aligned} \tag{4.4}$$

where

$$f_\mu = \nabla^\alpha h_{\mu\alpha}. \tag{4.5}$$

The terms in the second line involve the background Ricci tensor. It vanishes by the unperturbed Einstein's equation if the background is source-free as in previous vacuum case. But now in the radiation dominated background (2.9) these terms contribute, we shall denote them by $2Z_{\mu\nu}$. The terms in the first line have already been calculated in ref.[3], they will be written with a tilde $\tilde{G}$. The covariant derivatives refer to the background metric $g_{0\mu\nu}$.

To separate the angular dependence in the perturbed Einstein's equations

$$\delta G_{\mu\nu} = \frac{8\pi G}{c^4} \delta t_{\mu\nu} \tag{4.6}$$

we again use the Regge-Wheeler gauge [6] where $h_{\mu\nu}$ is of the form

$$h_{\mu\nu} = \begin{pmatrix} -c^2 H_2 & X H_1 & 0 & 0 \\ X H_1 & -X^2 H_0 & 0 & 0 \\ 0 & 0 & R^2 K & 0 \\ 0 & 0 & 0 & R^2 K \sin^2 \vartheta \end{pmatrix} Y_l^m(\vartheta.\phi). \tag{4.7}$$

Here Y_l^m denote the spherical harmonics, the functions H_0, H_1, H_2 and K depend on t and r only. Then the f_μ (4.5) are given by

$$\begin{aligned} f_0 &= \Big[-\partial_0 H_2 - \frac{1}{X} \partial_1 H_1 - \frac{\dot{X}}{X} H_0 - \Big(\frac{\dot{X}}{X} + 2\frac{\dot{R}}{R} \Big) H_2 + 2\frac{\dot{R}}{R} K \Big] Y \\ f_1 &= \Big[\frac{X}{c^2} \partial_0 H_1 + \partial_1 H_0 + \frac{2}{c^2} \Big(\dot{X} + X\frac{\dot{R}}{R} \Big) H_1 \Big] Y \\ f_2 &= -K\partial_2 Y, \quad f_3 = -K\partial_3 Y. \end{aligned} \tag{4.8}$$

To determine the right side in (4.6) we consider the background energy-momentum tensor (2.9) as a special case of the perfect fluid form

$$t_{\mu\nu} = (\varrho + p) u_\mu u_\nu - p g_{\mu\nu}$$

with $\varrho = \varrho_0$, $p = \varrho_0/3 = p_0$ and $u_\mu = (c, 0, 0, 0) = u_{0\mu}$. Here $\varrho_0(t)$ is the background radiation energy density (2.13), from now one we use the subscript 0 for the background. The nonspherical perfect fluid perturbations have been studied in [7]. The (polar) perturbations consist of a density perturbation

$\delta\varrho$, an entropy perturbation or a pressure perturbation δp, and a 3-velocity perturbation. Adapted to the Regge-Wheeler gauge (4.7) we only consider radial velocity perturbations δu_1. Then the perturbed energy-momentum tensor is given by

$$\delta t_{\mu\nu} = \begin{pmatrix} c^2\delta\varrho & c(\varrho_0+p_0)\delta u_1 & 0 & 0 \\ c(\varrho_0+p_0)\delta u_1 & -X^2\delta p & 0 & 0 \\ 0 & 0 & -R^2\delta p & 0 \\ 0 & 0 & 0 & -R^2\delta p\sin^2\vartheta \end{pmatrix} Y_l^m(\vartheta,\phi) - p_0 h_{\mu\nu}. \tag{4.9}$$

Note that $\delta u_0 = 0$ by normalization $u_\mu u^\mu = c^2$.

From the vanishing non-diagonal elements in (4.9) we get two homogeneous linear equations in (4.6). These equations are of the same form as in ref.[3], however, the equation for δG_{01} is lacking because $\delta t_{01} \neq 0$. From $\delta G_{23} = 0$ we obtain ([1],(3.2))

$$2\delta G_{23} = (\cot\vartheta\partial_3 - \partial_3\partial_2)(H_0 - H_2)Y = 0. \tag{4.10}$$

This yields the relation

$$H_0(t,r) = H_2(t,r). \tag{4.11}$$

Next from $\delta G_{02} = 0$ we get ([1],(3.4))

$$2\delta G_{02} = \Big[\partial_0(K-H_0) - \frac{1}{X}\partial_1 H_1 + H_0\Big(\frac{\dot R}{R} - \frac{\dot X}{X}\Big) - H_2\Big(\frac{\dot R}{R} + \frac{\dot X}{X}\Big)\Big]\partial_2 Y = 0.$$

For $l > 1$ this yields the first evolution equation

$$\partial_0(K - H_2) = -\frac{q^2}{X}H_3 + 2\frac{\dot X}{X}H_2. \tag{4.12}$$

Here we have used (4.11) and Fourier transformed quantities

$$\hat f(R,q) = (2\pi)^{-1/2}\int f(R,r)e^{-iqr}dr. \tag{4.13}$$

The function $H_3(t,q)$ stands for

$$H_3 = \frac{H_1}{iq}. \tag{4.14}$$

From $\delta G_{03} = 0$ the same equation is obtained.

The second evolution equation follows from ([3], (3.8))

$$2\delta G_{12} = [c^2\partial_1(K+H_2) + X\partial_0 H_1 + 2\dot X H_1]\partial_2 Y = 0. \tag{4.15}$$

After Fourier transformation this gives

$$X\partial_0 H_1 + ic^2 q(K+H_2) + 2\dot X H_1 = 0.$$

The same equation is obtained from $\delta G_{13} = 0$. We shall use this equation in the form

$$\partial_0 H_3 = -\frac{c^2}{X}(K+H_2) - 2\frac{\dot X}{X}H_3. \tag{4.16}$$

The calculation of

$$2\delta G_{01} = -\nabla^\alpha\nabla_\alpha h_{01} + \nabla_1 f_0 + \nabla_0 f_1 - 2R^\beta_{0\alpha 1}h^\alpha_\beta - \nabla_1\nabla_0 h^\alpha_\alpha +$$

$$+ R^\alpha_1 h_{0\alpha} + R^\alpha_0 h_{1\alpha} - R^\alpha_\alpha h_{01} + g_{01}h_{\alpha\beta}R^{\alpha\beta}. \tag{4.17}$$

requires important changes. The computation of

$$c^2\nabla_\alpha\nabla^\alpha h_{01} = X\partial_0^2 H_1 + \Big(\dot X + 2\frac{\dot R}{R}X\Big)\partial_0 H_1 - \frac{c^2}{X}\partial_1^2 H_1 -$$

$$-2c^2\frac{\dot{X}}{X}\partial_1(H_0+H_2)+c^2\frac{X}{R^2}l(l+1)H_1-\Big(4\frac{\dot{X}^2}{X}+2\frac{\dot{R}^2}{R^2}X\Big)H_1 \tag{4.18}$$

goes as in [3] apart from the factors c^2. Other pieces are

$$\nabla_1 f_0=-\partial_1\partial_0 H_2-\frac{1}{X}\partial_1^2 H_1-2\frac{\dot{X}}{X}\partial_1 H_9-\partial_1 H_2\Big(\frac{\dot{X}}{X}+2\frac{\dot{R}}{R}\Big)+$$

$$+2\frac{\dot{R}}{R}\partial_1 K-\frac{\dot{X}}{c^2}\partial_0 H_1-\frac{2}{c^2}\Big(\frac{\dot{X}^2}{X^2}+\dot{X}\frac{\dot{R}}{R}\Big)H_1 \tag{4.19}$$

$$\nabla_0 f_1=\frac{X}{c^2}\partial_0\check{s}H_1+\partial_0\partial_1 H_0+\frac{2}{c^2}\Big(\ddot{X}+X\frac{\ddot{R}}{R}-X\frac{\dot{R}^2}{R^2}-\frac{\dot{X}^2}{X}\Big)H_1+$$

$$+\frac{2}{c^2}\Big(\dot{X}+X\frac{\dot{R}}{R}\Big)\partial_0 H_1-\frac{\dot{X}}{X}\partial_1 H_0.$$

The new terms in the second line of (4.17) yield

$$\frac{2}{Y}Z_{01}=2R^1_1 h_{01}-R^\mu_\mu H_{01}=\frac{2}{c^2}XH_1\Big(2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R\check{s}}+\frac{c^2}{R^2}\Big). \tag{4.20}$$

This result can be simplified by means of Einstein's equations (2.10-12) for the background which we now write in the form

$$\begin{aligned}\frac{\ddot{R}}{R}&=-\frac{1}{2}\Big(\frac{\dot{R}^2}{R^2}+\frac{c^2}{R^2}+\frac{G_3}{3}\varrho_0\Big)\\ \frac{\dot{R}\dot{X}}{RX}&=-\frac{1}{2}\Big(\frac{\dot{R}^2}{R^2}+\frac{c^2}{R^2}+G_3\varrho_0\Big)\\ \frac{\ddot{X}}{X}&=\frac{\dot{R}^2}{R^2}+\frac{c^2}{R^2}-\frac{2}{3}G_3\varrho_0\end{aligned} \tag{4.21}$$

where

$$G_3=\frac{8\pi G}{c^2}. \tag{4.22}$$

Then the bracket in (4.21) simply becomes $-G_3\varrho_0/3$. This shows again that in the vacuum case $\varrho_0=0$ these terms do not contribute.

Now we are ready to write down the total Einstein's equation for δG_{01}. We obtain

$$\frac{\delta G_{01}}{Y}=\partial_1\partial_0 K-\frac{\dot{R}}{R}\partial_1 H_2+\Big(\frac{\dot{R}}{R}-\frac{\dot{X}}{X}\Big)\partial_1 K-H_1\Big[X\frac{l(l+1)}{2R^2}+\frac{X}{c^2}\Big(\frac{\ddot{R}}{R}-\frac{\dot{R}\dot{X}}{RX}\Big)\Big].$$

Using (4.21) again we get $G_3\varrho_0/3$ in the square bracket which compensates the new term Z_{01} (4.20). Then we totally have

$$\frac{\delta G_{01}}{Y}=\partial_1\partial_0 K-\frac{\dot{R}}{R}\partial_1 H_2+\Big(\frac{\dot{R}}{R}-\frac{\dot{X}}{X}\Big)\partial_1 K-X\frac{l(l+1)}{2R^2}H_1$$

$$=\frac{4}{3}\frac{G_3}{c^2}\varrho_0\delta u_1. \tag{4.23}$$

After radial Fourier transform (4.13) each derivative ∂_1 gives a factor iq. Using H_3 (4.14) this can be divided out on the left side. This shows that for a nontrivial solution the radial velocity δu_1 must have a potential

$$\delta u_1=\partial_1\Phi. \tag{4.24}$$

Then we obtain the following third evolution equation

$$\partial_0 K=\frac{\dot{R}}{R}H_2+\Big(\frac{\dot{X}}{X}-\frac{\dot{R}}{R}\Big)K++X\frac{l(l+1)}{2R^2}H_3+\frac{4}{3}\frac{G_3}{c^2}\varrho_0\Phi. \tag{4.25}$$

But now we need a forth evolution equation for the velocity potential Φ. This is derived in the next section from the diagonal components of the perturbed Einstein's equation (4.6). Using (4.25) in (4.12) we get

$$\partial_0 H_2 = \Big(\frac{\dot R}{R} - 2\frac{\dot X}{X}\Big) H_2 + \Big(\frac{\dot X}{X} - \frac{\dot R}{R}\Big) K + \\ + H_3\Big(\frac{q^2}{X} + X\frac{l(l+1)}{2R^2}\Big) + \frac{4}{3}\frac{G_3}{c^2}\varrho_0 \Phi. \tag{4.26}$$

Until now we have worked out the non-diagonal elements of the perturbed Einstein's equations. To get a closed system of evolution equations for all perturbations we must now investigate the diagonal elements also.

5 Evolution of the Perturbations

For the diagonal components we need various large pieces which are given in the appendix. We first calculate δG_{11} starting with

$$c^2\nabla_\alpha\nabla^\alpha h_{11} = -X^2\partial_0^2 H_0 + c^2\partial_1^2 H_0 - \Big(X\dot X + 2X^2\frac{\dot R}{R}\Big)\partial_0 H_0 + \\ + 4\dot X\partial_1 H_1 + 2\dot X^2 H_2 + H_0\Big(2\dot X^2 - c^2 X^2\frac{l(l+1)}{R^2}\Big). \tag{5.1}$$

With

$$c^2\nabla_1 f_1 = X\partial_1\partial_0 H_1 + c^2\partial_1^2 H_0 + (3\dot X + 2X\frac{\dot R}{R})\partial_1 H_1 + \\ + X\dot X\partial_0 H_2 + \dot X^2 H_0 + (\dot X^2 + 2X\dot X\frac{\dot R}{R})H_2 - 2X\dot X\frac{\dot R}{R}K \tag{5.2}$$

we finally obtain

$$2c^2\frac{\delta\tilde G_{11}}{X^2 Y} = 2\partial_0^2 K - 2\frac{\dot R}{R}\partial_0 H_2 + 6\frac{\dot R}{R}\partial_0 K - 2\Big(\frac{\ddot R}{R} + \frac{\dot R^2}{R^2}\Big)H_2 - \\ - 2\frac{\dot R\dot X}{RX}H_0 + 2K\Big(\frac{\ddot R}{R} + \frac{\dot R^2}{R^2} + \frac{\dot R\dot X}{RX}) + c^2\frac{l(l+1)}{R^2}(H_2 + K). \tag{5.3}$$

In case of the vacuum background this vanishes if the evolution equations are taken into account. Therefore we now also differentiate (4.25) and substitute the first derivatives by (4.16), (4.25) and (4.26) yielding

$$\partial_0^2 K = K\Big(\frac{\ddot X}{X} - \frac{\ddot R}{R} - \frac{\dot R\dot X}{RX} + \frac{\dot R^2}{R^2} - c^2\frac{l(l+1)}{2R^2} - G_3\frac{\varrho_0}{X}\Big) + \\ + H_2\Big(\frac{\ddot R}{R} - \frac{\dot R\dot X}{RX} - \frac{\dot R^2}{R^2} - c^2\frac{l(l+1)}{2R^2} - G_3\frac{\varrho_0}{X}\Big) + \\ H_3\Big(\frac{q^2}{X}\frac{\dot R}{R} - X\frac{\dot R}{R}\frac{l(l+1)}{R^2} - \frac{G_3}{c^2}\varrho_0\frac{\dot X}{X} + \frac{G_3}{c^2}\dot\varrho_0\Big) + \frac{4}{3}\frac{G_3}{c^2}\Big(\varrho_0\Phi\frac{\dot X}{X} + \dot\varrho_0\Phi + \varrho_0\dot\phi\Big). \tag{5.4}$$

Now we are ready to simplify (5.3). Substituting all temporal derivatives by the expressions without derivatives we see a great cancellation:

$$c^2\frac{\delta\tilde G_{11}}{X^2 Y} = \frac{G_3}{c^2}\varrho_0\Phi\Big[\frac{4}{3}\frac{\dot X}{X} + \frac{8}{3}\frac{\dot R}{R}\Big) + \\ + \frac{4}{3}\frac{G_3}{c^2}\Big[\varrho_0\partial_0\Phi - \frac{4}{3}\Phi\Big(\frac{\dot X}{X} + 2\frac{\dot R}{R}\Big)\Big] + K\Big(\frac{\ddot X}{X} + 2\frac{\dot R\dot X}{RX}\Big). \tag{5.5}$$

Here we can use (4.21) in the last term, then the coefficient of K becomes $G_3\varrho_0/3$ which is proportional to ϱ_0, too, as it must be. For the total δG_{11} we must add the additional terms from the second line of (4.4)

$$\frac{Z_{11}}{Y} = \frac{G_3}{2c^2}\varrho_0 X^2(H_2 - \frac{K}{3}). \tag{5.6}$$

Using energy conservation

$$\dot{\varrho}_0 = -\frac{4}{3}\varrho_0\Big(\frac{\dot{X}}{X}+2\frac{\dot{R}}{R}\Big) \tag{5.7}$$

we end up with

$$c^2\frac{\delta G_{11}}{X^2Y} = G_3\varrho_0\Big[H_2 - 2K\Big)+\frac{4}{3}\frac{G_3}{c^2}\varrho_0\partial_0\Phi-$$
$$-\frac{4}{9}\frac{G_3}{c^2}\varrho_0\Phi\Big(\frac{\dot{X}}{X}+2\frac{\dot{R}}{R}\Big)= -G_3\delta p. \tag{5.8}$$

The last equality is Einstein's equation (4.6).

The calculation of δG_{22} is simpler. The various contributions are collected in the appendix. The final result without Z_{22} is given by

$$2c^2\frac{\delta\tilde{G}_{22}}{R^2Y} = \partial_0^2(K-H_2)-\frac{c^2}{X^2}\partial_1^2(K+H_2)-\frac{2}{X}\partial_0\partial_1H_1+$$
$$+\Big(2\frac{\dot{R}}{R}+\frac{\dot{X}}{X}\Big)\partial_0K-\Big(\frac{\dot{R}}{R}+\frac{\dot{X}}{X}\Big)\partial_0H_2-2\Big(\frac{\dot{X}}{X^2}+\frac{\dot{R}}{RX}\Big)\partial_1H_1-$$
$$-\Big(2\frac{\dot{X}}{X}+\frac{\dot{R}}{R}\Big)\partial_0H_0-\Big(\frac{\ddot{X}}{X}+2\frac{\dot{R}\dot{X}}{RX}\Big)(H_2+H_0)+$$
$$+2K\Big(\frac{\ddot{R}}{R}+\frac{c^2}{R^2}+\frac{\dot{R}^2}{R^2}+\frac{\dot{R}\dot{X}}{RX}\Big). \tag{5.9}$$

Using $H_0 = H_2$ and going over to the Fourier transformed quantities we find

$$2c^2\frac{\delta\tilde{G}_{22}}{R^2Y} = \partial_0^2(K-H_2)+c^2\frac{q^2}{X^2}(K+H_2)+2\frac{q^2}{X}\partial_0H_3+$$
$$+\Big(\frac{\dot{X}}{X}+2\frac{\dot{R}}{R}\Big)\partial_0K-\Big(3\frac{\dot{X}}{X}+2\frac{\dot{R}}{R}\Big)\partial_0H_2+2q^2\Big(\frac{\dot{X}}{X^2}+\frac{\dot{R}}{RX}\Big)H_3-$$
$$-2\Big(\frac{\ddot{X}}{X}+2\frac{\dot{R}\dot{X}}{RX}\Big)H_2+2K\Big(\frac{\ddot{R}}{R}+\frac{c^2}{R^2}+\frac{\dot{R}^2}{R^2}+\frac{\dot{R}\dot{X}}{RX}\Big). \tag{5.10}$$

Here in the first term with ∂_0^2 we substitute the difference of (4.25) and (4.26) and we express all first derivatives by the previous formulas. Then again there is a huge cancellation which is expected because in the vacuum case the result is 0. By (4.21) we now get the simple result

$$2c^2\frac{\delta\tilde{G}_{22}}{R^2Y} = -\frac{4}{3}G_3K\varrho_0.$$

Adding the additional term Z_{22} we arrive at

$$c^2\frac{\delta G_{22}}{R^2Y} = \frac{G_3}{3}\varrho_0(H_2-3K) = G_3\delta p \tag{5.11}$$

where again the last equality is Einstein's equation. The simple result (5.11) for the pressure perturbation can be substituted into (5.8). Then we obtain the desired evolution equation for the velocity potential Φ:

$$\partial_0\Phi = \frac{1}{3}\Big(\frac{\dot{X}}{X}+2\frac{\dot{R}}{R}\Big)\Phi+\Big(\frac{9}{4}K-H_2)c^2. \tag{5.12}$$

In principle we now have obtained four evolution equations (4.16), (4.25-26) and (5.12) for all perturbed modes. However to integrate the equations from the present time backwards to the Big Bang we need four initial conditions. The best measured quantities for this purpose are the CMB anisotropies which according to (5.11) give $H_2 - 3K$ because this is known from the radiation pressure δp. To find K and H_2 separately we must investigate δG_{00} which yields the total energy density (radiation plus matter).

Appendix

The following results are needed for all diagonal components of the perturbed Einstein's equations:

$$2c^2\frac{\nabla^\alpha f_\alpha}{Y} = -\partial_0^2 H_2 - \frac{c^2}{X^2}\partial_1^2 H_0 - \frac{2}{X}\partial_0\partial_1 H_1 - \partial_1 H_1\Big(2\frac{\dot{X}}{X}+4\frac{\dot{R}}{R}\Big)-$$

$$-2\partial_0 H_2\Big(\frac{\dot{X}}{X}+2\frac{\dot{R}}{R}\Big)-\frac{\dot{X}}{X}\partial_0 H_0 + 2\frac{\dot{R}}{R}\partial_0 K - H_0\Big(\frac{\ddot{X}}{X}+2\frac{\dot{R}\dot{X}}{RX}\Big)+$$

$$+K\Big(2\frac{\ddot{R}}{R}+2\frac{\dot{R}^2}{R^2}+2\frac{\dot{R}\dot{X}}{RX}-c^2\frac{l(l+1)}{R^2}\Big)-H_2\Big(\frac{\ddot{X}}{X}+2\frac{\ddot{R}}{R}+2\frac{\dot{R}\check{s}}{R^2}+4\frac{\dot{R}\dot{X}}{RX}\Big). \quad (A.1)$$

$$\nabla_\mu\nabla^\mu h_\alpha^\alpha = \Big(\frac{1}{c^2}\partial_0^2 - \frac{1}{X^2}\partial_1^2 + \frac{l(l+1)}{R^2}\Big)(H_0 - H_2 - 2K)Y+$$

$$+\frac{1}{c^2}\Big(\frac{\dot{X}}{X}+2\frac{\dot{R}}{R}\Big)\partial_0(H_0 - H_2 - 2K)Y, \quad (A.2)$$

so that

$$\frac{c^2}{Y}\Big(\nabla^\alpha f_\alpha - \nabla^\beta\nabla_\beta h_\alpha^\alpha\Big)= \partial_0^2(2K - H_0) - \frac{c^2}{X^2}\partial_1\check{s}(2K - H_2) - \frac{2}{X}\partial_0\partial_1 H_1-$$

$$-\Big(2\frac{\dot{X}}{X^2}+4\frac{\dot{R}}{XR}\Big)\partial_1 H_1 - \Big(2\frac{\dot{X}}{X}+2\frac{\dot{R}}{R}\Big)\partial_0 H_0 - \Big(\frac{\dot{X}}{X}+2\frac{\dot{R}}{R}\Big)\partial_0 H_2+$$

$$+\Big(2\frac{\dot{X}}{X}+6\frac{\dot{R}}{R}\Big)\partial_0 K - \Big(\frac{\ddot{X}}{X}+2\frac{\dot{X}\dot{R}}{XR}\Big)H_0 - \Big(\frac{\ddot{X}}{X}+2\frac{\ddot{R}}{R}+2\frac{\dot{R}^2}{R^2}+4\frac{\dot{X}\dot{R}}{XR}\Big)H_2+$$

$$+\Big(2\frac{\ddot{R}}{R}+2\frac{\dot{R}^2}{R^2}+2\frac{\dot{X}\dot{R}}{XR}\Big)K + c^2\frac{l(l+1)}{R^2}(H_2 - H_0 + K). \quad (A.4)$$

For δG_{22} we need

$$\frac{c^2}{R^2}\nabla_\alpha\nabla^\alpha h_{22} = \partial_0^2 K + \Big(2\frac{\dot{R}}{R}+\frac{\dot{X}}{X}\Big)\partial_0 K - \frac{c^2}{X^2}\partial_1^2 K+$$

$$+c^2\frac{l(l+1)}{R^2}K + 2\frac{\bar{R}^2}{R\check{s}}H_2 - 2\frac{\dot{R}\check{s}}{R^2}K \quad (A.5)$$

and

$$\partial_2 f_2 = \partial_2 f_2 - \mathit{\Gamma}_{22}^0 f_0 =$$

$$= -K\partial_2^2 Y - \frac{R\dot{R}}{c^2}\Big[-\partial_0 H_2 - \frac{1}{X}\partial_1 H_1 - \frac{\dot{X}}{X}(H_0 + H_2) + 2\frac{\dot{R}}{R}(K - H_2)\Big]Y. \quad (A.6)$$

From the Riemann tensor we get

$$-2R_{202}^0 H_0 = 2\frac{R}{c^2}\ddot{R}H_2 Y$$

$$-2R_{212}^1 H_0 = -2\frac{R}{c^2}\frac{\dot{R}\dot{X}}{X}H_0 Y \quad (A.7)$$

In addition we have

$$\nabla_2\nabla_2 h_\alpha^\alpha = \partial_2^2 h_\alpha^\alpha - \mathit{\Gamma}_{22}^0\partial_0 h_\alpha^\alpha =$$

$$= (H_0 - H_2 - 2K)\partial_2^2 Y - \frac{R\dot{R}}{c^2}\partial_0(H_0 - H_2 - 2K). \quad (A.8)$$

Obituary

Prof. Dr. Günter Robert Scharf was born on September 19, 1938, in Nordhausen, Germany. He died on August 16, 2020, near Zürich, Switzerland, after a short illness due to a fall at home.

He was my esteemed doctoral adviser at the University of Zürich.

Günter enrolled in 1958 at the University of Göttingen to study physics, and continued his studies at the University of Giessen in the following year. The fact that Scharf lost a leg in a tragic motorcycle accident never stopped him from continuing his scientific career; thanks to a foreign exchange scholarship, Scharf was able to continue his studies at the Swiss Federal Institute of Technology in Zürich (ETHZ), where he wrote his diploma thesis under the supervision of Prof. Dr. Res Jost in 1962.

In 1965, Günter finished his Ph.D. thesis *Fastperiodische Potentiale* under the supervision of Prof. Dr. Armin Thellung (1924-2003), who was one of the last Ph.D. students and assistants of Wolfgang Pauli.

Günter Scharf has written three excellent books:

– *Finite Quantum Electrodynamics* (Springer, 1989/1995; Dover, 2014), in which he shows how one can avoid ultraviolet divergences in QED by making use of causality and distribution theory.
– *Quantum Gauge Theories : A True Ghost Story* (Wiley, 2001; Dover 2016). In this book, the causal method is extended to gauge theories.
– *From Electrostatics to Optics* (Springer, 1994) is an excellent textbook containing a concise introduction to classical electrodynamics.

The causal approach to quantum field theory advocated during the last 35 years by Günter Scharf as his major research interest goes back to a classic paper by Henri Epstein and Vladimir Glaser [9]. The method has the great advantage that it uses mathematically well-defined objects only, namely free asymptotic fields. Therefore all mathematical operations have a precise meaning in the framework of distribution theory, in particular, there are no ultraviolet divergences. The method has been applied to abelian, massless non-abelian and to massive non-abelian gauge theories. In the latter case one obtains the complete structure of the standard electroweak theory as a consequence of (quantum) gauge invariance, without relying explicitly on the concept of spontaneous symmetry breaking. In the case of spin-2 gauge fields on a flat background gauge invariance alone leads to the same couplings as given by Einstein's theory.

Andreas Aste
September 9, 2020
Basel, Switzerland

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