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Newton-type Methods for Solving the Nonsmooth Equations with Finitely Many Maximum Functions

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Abstract In this paper, we consider the Newton type methods for solving the nonsmooth equations with finitely many maximum functions. A new ∂_*-differential is used in the given Newton-type methods. The Newton-type methods also include a new parameterized combination. The superlinear convergence of the given methods is presented. Finally, the numerical experiments highlight the efficiency of the given methods.

Keywords: Nonsmooth equations, parameterized combinations, Newton method, superlinear convergence.

1 Introduction

In this paper, we consider the nonsmooth equations with finitely many maximum functions

$$\begin{aligned} \max_{j\in J_1} f_{1j}(x) &= 0, \\ &\vdots \\ \max_{j\in J_n} f_{nj}(x) &= 0, \end{aligned} \tag{1.1}$$

where $f_{ij}: R^n \to R$ for $j \in J_i, i = 1, \cdots, n$ are assumed to be continuous differentiable, and $J_i, i = 1, \cdots, n$ are finite index sets. Therefore, we know that (1.1) is a system of semismooth equations. As in [1], throughout the whole paper, we denote

$$f_i(x) = \max_{j\in J_i} f_{ij}(x), x \in R^n, i = 1, \cdots, n,$$

$$F(x) = (f_1(x), \cdots, f_n(x))^T, x \in R^n,$$

$$J_i(x) = \{j \in J_i | f_{ij}(x) = f_i(x)\}, x \in R^n, i = 1, \cdots, n.$$

And (1.1) can be transformed into the nonsmooth equations

$$F(x) = 0. \tag{1.2}$$

Obviously, F is locally Lipschitzian and semismooth on R^n, see [1]. Many optimization problems and mathematical programming problems can be briefly rewritten as the form (1.2), e.g., LC^1 optimization problems([2-5]), nonlinear complementarity problems([6]) and variational inequality problems([7]).

Nonsmooth equation (1.2) was considered concurrently in [2] and [8]. One of the fundamental versions of the generalized Newton method was proposed by Qi and Sun in [2] as

$$x_{k+1} = x_k - V_k^{-1} F(x_k),$$

where V_k is an element of Clarke generalized Jacobian [2], an element of B-differential [3] and an element of b-differential [9] of F at x_k. And in [5], Chen and Qi presented the parameterized modification of generalized Jacobian Newton-like method

$$x_{k+1} = x_k - \alpha_k (V_k + \lambda_k I)^{-1} F(x_k),$$

where $V_k \in \partial_B F(x_k)$, I is the $n \times n$ identity matrix, α_k and λ_k are chosen to ensure convergence and $V_k + \lambda_k I$ is invertible.

Recently, some research works have been conducted for solving (1.1). Specifically, Gao in [10] considered the Newton method for solving the problem (1.1), which is widely used in the optimal control, the variational inequality and complementarity problems, equilibrium problems, engineering mechanics [8,11,12] and Karush-Kuhn-Tucker systems of nonlinear programming problems. Śmietański in [6] constructed the difference approximated Jacobian for a finite maximum function. A new smoothing nonlinear conjugate gradient method and a modified Levenberg-Marquardt method were also proposed for solving the nonsmooth equations (1.1) in [13] and [14]. Moreover, Śmietański considered the midpoint generalized Newton method in [15], in which a new approximation x^{k+1} is

$$x^{k+1} = x^k - (V_{xz}^k)^{-1} F(x^k), k = 0, 1, \cdots,$$

where V_{xz}^k is an element of some subdifferential of F at $\frac{1}{2}(x^k + z^k)$ and $z^k = x^k - (V_x^k)^{-1} F(x^k), V_x^k \in \partial_B F(x)$.

Based on the above research work, we present two new Newton-type methods with ∂_*-differential for (1.1). The ∂_*-differential for (1.1) is defined as

$$\partial_* F(x) = \{(\nabla f_{1j_1}, \cdots, \nabla f_{nj_n})^T | j_1 \in J_1(x), \cdots, j_n \in J_n(x)\}, x \in R^n. \tag{1.3}$$

And the $\partial_* F(x)$ is a non-empty bounded set for each x such that

$$\partial_* F(x) \subset \partial f_1(x) \times \cdots \times \partial f_n(x),$$

where $\partial f_i(x)$ is the Clarke generalized gradient of f_i at x.

The remainder of the paper is organized as follows. In Section 2, we recall some proverbial results of generalized Jacobian, semismoothness and some propositions. In Section 3, we present a parameterized combinations Newton method for solving (1.1) and give the superlinear convergence of it. A modified parameterized combinations Newton method is also given to solve (1.1) and the local superlinear convergence is also proved. In Section 4, we report some numerical results of the two new Newton-type methods. In Section 5, we give some discussions to conclude this paper.

2 Preliminaries

If $F(x) : R^n \to R^n$ is a locally Lipschitz function, and the limit

$$\lim_{\substack{V \in \partial F(x + th') \\ h' \to h, t \to 0^+}} Vh'$$

exists for any $h \in R^n$, then we say that F is semismooth at x. As we all know, piecewise smooth functions and the maximum of a finite number of smooth functions are semismooth, which are introduced in [2,10,15,16].

Proposition 2.1. *Let $x \in R^n$. Suppose that for any $h \in R^n$*

$$\lim_{V \in \partial F(x+th), t \downarrow 0} Vh$$

exists. Then

$$F'(x; h) = \lim_{V \in \partial F(x+th), t \downarrow 0} Vh.$$

Proposition 2.2. *F is semismooth at x.* $\Longleftrightarrow$

$$Vh - F'(x; h) = o(\|h\|). \tag{2.1}$$

From [2], for any $h \to 0$,

$$F(x+h) - F(x) - F'(x;h) = o(\|h\|), \tag{2.2}$$

then we say F is semismooth at x.

As [1], we give the following proposition.

Proposition 2.3. *Suppose that F is defined by (1.2). Then, for any $x, x^* \in R^n, V \in \partial F(x)$, we get*

$$F(x) - F(x^*) - V(x - x^*) = o(\|x - x^*\|).$$

Proof. By (2.1) and (2.2), we can obtain

$$Vh - F'(x^*;h) = o(\|h\|), \forall V \in \partial F(x^* + h), \tag{2.3}$$

$$F(x^* + h) - F(x^*) - F'(x^*;h) = o(\|h\|). \tag{2.4}$$

Let $h = x - x^*$, then we obtain the proposition. □

3 The Newton-type Methods and Their Superlinear Convergence

In this section, we present the new parameterized Newton methods for solving (1.1). Now we give the framework of the parameterized combinations Newton method.

Method 3.1(Parameterized Combinations Newton Method)

Step 0. Let $0 \le \epsilon \le 1, \alpha, \lambda \in (0,1), \lambda_1, \lambda_2 \in [0,1], \lambda_1 + \lambda_2 = 1$, $x^0 \in R^n$. Let $k := 0$.

Step 1. If $\|F(x^k)\| \le \epsilon$, then STOP.

Step 2. Find a point $y^k \in R^n$ such that

$$y^k = x^k - \alpha(V_x^k + \lambda I)^{-1} F(x^k),$$

where $V_x^k \in \partial_* F(x^k)$.

Step 3. Set

$$x^{k+1} = x^k - (V_{xy}^k)^{-1} F(x^k), k = 0, 1, \cdots, \tag{3.1}$$

where $V_{xy}^k \in \partial_* F(\lambda_1 x^k + \lambda_2 y^k)$. Let $k := k + 1$, go to Step 1.

Remark 3.1. *If $\lambda_1 = \lambda_2 = \frac{1}{2}$, the method we presented is equal to the case in [15].*

Below we prove the convergence results of Method 3.1 with the ∂_*-differential. First of all, we need some helpful lemmas.

Lemma 3.2. *Suppose that $F(x)$ and $\partial_* F(x)$ are defined by (1.2) and (1.3) respectively, and for any V, $V \in \partial_* F(x)$ is nonsingular. Then there exist $C > 0, \gamma > 0, \epsilon > 0$ and $N(x, \epsilon)$ is a neighbor of x, such that*

$$\|V^{-1}\| \le C, \forall V \in \partial_* F(x), x \in N(x, \epsilon), \tag{3.2}$$

$$\|V\| \le \gamma, \forall V \in \partial_* F(x), x \in N(x, \epsilon).$$

Lemma 3.3. *Suppose that x^* is the solution of (1.1). Then for any constants $\alpha, \lambda, \lambda_2 \in (0,1)$, the function $G(x) = x - (\overline{V}_x)^{-1} F(x)$, where $\overline{V}_x \in \partial_* F(x - \alpha\lambda_2(V_x + \lambda I)^{-1} F(x))$ and $V_x \in \partial_* F(x)$, is well-defined in a neighborhood of x^*.*

Proof. By Lemma 3.1, there exists a scalar $C > 0$ and a neighborhood N of x^* such that V_x is nonsingular and $\|V_x^{-1}\| \le C$ for any $x \in N(x, \epsilon)$ and $V_x \in \partial_* F(x)$. Firstly, let $\epsilon \in (0, \frac{1}{2C})$, we know that there exists a $V_{x^*} \in \partial_* F(x^*)$ such that for any $x \in S(x^*, \delta_1)$ and $V_x \in \partial_* F(x)$,

$$\|V_x - V_{x^*}\| < \epsilon. \tag{3.3}$$

Then, we consider $y = x - \alpha\lambda_2(V_x + \lambda I)^{-1} F(x)$, where $x \in N(x, \epsilon)$, $V_x \in \partial_* F(x)$ and $\alpha, \lambda, \lambda_2 \in (0,1)$. By the Corollary 3.2 in [3], it can be guaranteed that $y \in S(x^*, \delta_1)$, where $S(x^*, \delta_1)$ is an open ball in R^n with center x^* and radius δ_1. Since $V_y \in \partial_* F(y)$, then (3.3) holds, i.e.

$$\|V_y - V_{x^*}\| < \epsilon.$$

So, let $\delta = min\{\epsilon, \delta_1\}$, using the Banach perturbation lemma in [17], we obtain that V_x is nonsingular and

$$\begin{aligned}\|(\overline{V}_x)^{-1}\| &= \|(V_y)^{-1}\| = \|[V_{x^*} + (V_y - V_{x^*})]^{-1}\| \\ &\leq \frac{\|(V_{x^*})^{-1}\|}{1 - \|(V_{x^*})^{-1}\|\|V_y - V_{x^*}\|} \\ &\leq \frac{C}{1 - C\epsilon} \leq 2C\end{aligned}$$

for $x \in S(x^*, \delta)$. Then, we get it. □

Theorem 3.4. *Suppose that x^* is a solution of (1.1) and $\|V_{x^*}\| \leq \gamma$ for all $V_{x^*} \in \partial_* F(x^*)$ and all V_{x^*} are nonsingular. Then there exists a neighborhood of x^* such that the sequence $\{x^k\}$ generated by Method 3.1 with ∂_*-differential converges superlinearly to x^* for any initial point x^0 belonging to this neighborhood. Besides that, if $F(x^k) \neq 0$ for all k, then the norm of F decreases superlinearly in a neighborhood of x^*, i.e.*

$$\lim_{k\to\infty} \frac{\|F(x^{k+1})\|}{\|F(x^k)\|} = 0. \tag{3.4}$$

Proof. By Lemma 3.1 and Lemma 3.2, for the first step $k = 0$, the iterative formula (3.1) is well-defined in a neighborhood of x^*. And based on Lemma 3.2, we know that if $y = x - \alpha\lambda_2(V_x + \lambda I)^{-1}F(x)$, then we have $y \in S(x^*, \delta)$ and

$$\begin{aligned}\|x^{k+1} - x^*\| &= \|x^k - (V_{xy}^k)^{-1}F(x^k) - x^*\| \\ &= \|(V_{xy}^k)^{-1}\|\|F(x^k) - F(x^*) - V_{xy}^k(x^k - x^*)\| \\ &= o(\|x^k - x^*\|),\end{aligned}$$

where $V_{xy}^k \in \partial_* F(\lambda_1 x^k + \lambda_2 y^k), \lambda_1 + \lambda_2 = 1$ and $\lambda_1, \lambda_2 \in [0, 1]$. Then the sequence $\{x^k\}(k \in N)$ is superlinearly convergent to x^*.

In the following we give the proof of (3.4). By Lemma 3.1, there exist $C > 0$ and $\epsilon > 0$ such that V is nonsingular and $\|V^{-1}\| \leq C$ for any $x \in S(x^*, \epsilon)$ and $V \in \partial_* F(x)$. By Proposition 2.3, for any $\alpha \in (0, 1)$, there is a $\delta_2 \in (0, \epsilon)$ such that if $x \in S(x^*, \delta_2)$, we can see that

$$\|F(x) - V(x - x^*)\| \leq \alpha\|x - x^*\|. \tag{3.5}$$

And if $x^k \in S(x^*, \delta)$, we have

$$\|x^{k+1} - x^*\| \leq \alpha\|x^k - x^*\| \tag{3.6}$$

for $\delta \in (0, \delta_2)$. Since $\{x^k\}$ converges to x^*, there exists a $k_\delta \in N$ such that $\|x^k - x^*\| \leq \delta$ for all $k \geq k_\delta$. By (3.6), we have $\|x^{k+1} - x^*\| \leq \delta \leq \delta_2$. Furthermore, by (3.3), we have

$$\|V_{xy}^{k+1}\| = \|V_{xy}^{k+1} - V_{x^*} + V_{x^*}\| \leq \epsilon + \|V_{x^*}\| \leq \epsilon + \gamma.$$

By (3.5) and (3.6), we know that

$$\begin{aligned}\|F(x^{k+1})\| &\leq \|V_{xy}^{k+1}(x^{k+1} - x^*)\| + \alpha\|x^{k+1} - x^*\| \\ &\leq (\epsilon + \gamma + \alpha)\|x^{k+1} - x^*\| \\ &\leq \alpha(\epsilon + \gamma + \alpha)\|x^k - x^*\|.\end{aligned}$$

By (3.1),(3.2) and (3.6), we obtain that

$$\begin{aligned}\|x^k - x^*\| &\leq \|x^{k+1} - x^k\| + \|x^{k+1} - x^*\| \\ &\leq \|(V_{xy}^k)^{-1}F(x^k)\| + \alpha\|x^k - x^*\| \\ &\leq C\|F(x^k)\| + \alpha\|x^k - x^*\|.\end{aligned}$$

Then

$$\|x^k - x^*\| \le \frac{C}{1-\alpha}\|F(x^k)\|.$$

Then we get

$$\|F(x^{k+1})\| \le \alpha(\epsilon+\gamma+\alpha)\|x^k - x^*\| \le \frac{C\alpha(\epsilon+\gamma+\alpha)}{1-\alpha}\|F(x^k)\|.$$

Since $F(x^k) \neq 0$ for all k and α may be arbitrarily small as $k \to \infty$, we get (3.4). Hence, we get this theorem. □

Based on Method 3.1, in the following of this section, we establish a new iterator formula for solving (1.1). Given the kth approximation x^k, the modification of the parameterized combinations Newton method obtains x^{k+1} by means of

$$x^{k+1} = x^k - \beta_k(V_{xy}^k + \mu_k I)^{-1}F(x^k), k = 0, 1, \cdots, \tag{3.7}$$

where $V_{xy}^k \in \partial_* F(\lambda_1 x^k + \lambda_2 y^k), \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \in [0,1]$ is an element of some subdifferential of F at $\lambda_1 x^k + \lambda_2 y^k$ and $y^k = x^k - \alpha_k(V_x^k + \lambda_k I)^{-1}F(x^k)$, where $V_x^k \in \partial_* F(x^k)$.

Then we give the framework of the new modified method.

Method 3.2(Modified Newton Method)

Step 0. Let $0 \le \epsilon \le 1, \alpha, \beta, \lambda, \mu, \lambda_1, \lambda_2 \in [0,1], \lambda_1 + \lambda_2 = 1$, $x^0 \in R^n$. Let $k := 0$.

Step 1. If $\|F(x^k)\| \le \epsilon$, then STOP.

Step 2. Compute

$$y^k = x^k - \alpha(V_x^k + \lambda I)^{-1}F(x^k),$$

where $V_x^k \in \partial_* F(x^k)$.

Step 3. By (3.7), we obtain x^{k+1}. Let $k := k+1$, go to Step 1.

Remark 3.2. In Method 3.2, we replace the parameters β_k and μ_k which are defined by (3.7) by $\beta \in (0,1], \mu \in [0,1]$, respectively.

The following theorem is the important result in this section and it also shows the local superlinearly convergence of the Method 3.2.

Theorem 3.5. *Suppose that x^* is a solution of (1.1), and all $V_{x^*} \in \partial_* F(x^*)$ are nonsingular. Let ϵ, β_k, and μ_k satisfy $0 < \beta_1 < \beta_k < 1, \epsilon < \gamma, 0 < C(\epsilon(2+\beta_1) + \gamma(1-\beta_1)) < 1$ and*

$$|\mu_k| \le \mu' < \frac{1 - C(\epsilon(2+\beta_1) + \gamma(1-\beta_1))}{2C}.$$

Then there exists a scalar $\delta > 0$ such that for any $x^0 \in S(x^*, \delta)$, the sequence $\{x^k\}$ defined by Method 3.2 is well-defined and converges linearly to x^*. Furthermore, if $\beta_k \to 1$ and $\mu_k \to 0$ as $k \to \infty$, then $\{x^k\}$ converges superlinearly to x^*.

Proof. By (2.3) and (2.4), there exists a constant $\delta > 0$ such that for any $x \in S(x^*, \delta)$, $V_x \in \partial_* F(x)$ and $h = x - x^*$, we have

$$\|V_x(x - x^*) - F'(x^*; x - x^*)\| \le \epsilon\|x - x^*\|,$$
$$\|F(x) - F(x^*) - F'(x^*; x - x^*)\| \le \epsilon\|x - x^*\|.$$

By this formula together with (3.3), we obtain that

$$\|V_x + \mu I - V_{x^*}\| \le \epsilon + |\mu| < \frac{1}{C}$$

when $|\mu| < \mu'$. This implies that $V_x + \mu I$ is nonsingular and

$$\|(V_x + \mu I)^{-1}\| \le \frac{C}{1 - C(\epsilon + |\mu|)}.$$

Therefore (3.7) is well-defined for $x^* \in S(x^*, \delta)$. Furthermore, by (3.3) and $\|V_{xy}^k\| \leq \epsilon + \|V_{x^*}\| \leq \epsilon + \gamma$, we have

$$\begin{aligned}\|x^{k+1} - x^*\| &\leq \|(V_{xy}^k + \mu_k I)^{-1}\|(\beta_k \|F(x^k) - F(x^*) - F'(x^*; x^k - x^*)\| \\ &+ \beta_k \|V_{xy}^k(x^k - x^*) - F'(x^*; x^k - x^*)\| + ((1-\beta_k)\|V_{xy}^k\| + |\mu_k|)\|x^k - x^*\|) \\ &\leq \frac{C}{1 - C(\epsilon + |\mu_k|)}(2\beta_k + (1-\beta_k)(\epsilon + \gamma) + |\mu_k|)\|x^k - x^*\| \\ &\leq \rho \|x^k - x^*\|,\end{aligned}$$

where $0 < \rho = \frac{C}{1 - C(\epsilon + \mu')}(2\beta_1 \epsilon + (1-\beta_1)(\epsilon + \gamma) + \mu') < 1$. Hence $\{x^k\}$ defined by (3.7) converges linearly to x^*.

By

$$\|F(x) - F(x^*) - F'(x^*; x - x^*)\| = o\|x - x^*\|,$$

$$\|V_x(x - x^*) - F'(x^*; x - x^*)\| = o\|x - x^*\|,$$

and let $\beta_k \to 1, \mu_k \to 0$ as $k \to \infty$, we get

$$\|x^{k+1} - x^*\| = o(\|x^k - x^*\|).$$

Hence $\{x^k\}$ defined by (3.7) converges superlinearly to x^*, if $\beta_k \to 1, \mu_k \to 0$ as $k \to \infty$. □

4 Numerical Results

In this part, we give some numerical results to illustrate the efficiency of the two given Newton-type methods. The Example 4.1 is based on the Example 4.1 in [14]. We compared Method 3.1 with **fsolve** which is taken from the Matlab optimization tool box. And for **fsolve** we choose "Algorithm to Levenberg-Marquardt" and "Function tolerance to 1e-15". The numerical results are given in Table 4.1. The Example 4.3 and Example 4.4 are two semismooth equations from [15]. We compare the Method 3.1 and Method 3.2 with the midpoint Newton method [15], respectively, and give some numerical examples to illustrate the efficiency of the methods presented in our paper. The following notations are used: M1 is the Method 3.1; M2 is the Method 3.2; M3 is the midpoint Newton method introduced by Śmietański in [15]; x^0 is the initial point; N is the number of iterations; x^k is the final value after the iterative process. The symbol $-$ denotes the test has failed. And the parameters used in this section are $\alpha = 0.5, \beta = 0.85, \lambda = 0.5, \mu = 0.125, \lambda_1 = 0.6, \lambda_2 = 0.4$ for all tests.

Example 4.1.

$$\max\{f_{11}(x_1, x_2), f_{12}(x_1, x_2)\} = 0,$$

$$\max\{f_{21}(x_1, x_2), f_{22}(x_1, x_2)\} = 0,$$

where $f_{11} = x_1^2 + x_2^2, f_{12} = x_1^2, f_{21} = \frac{1}{2}(x_1 + 2x_2)^2, f_{22} = 2(x_1 + 2x_2)^2$. *From (1.1), we know*

$$F(x) = (f_1(x), f_2(x))^T,$$

where $f_1(x) = x_1^2 + x_2^2, f_2(x) = 2(x_1 + 2x_2)^2, x \in R^2$. *We know that* $F(x)$ *has one solution* $(0, 0)^T$. *Then we compare M1 with* ***fsolve*** *in the final solution. The computational results are summarized in Table 4.1.*

Table 4.1

		M1	*fsolve*
	x^0	x^k	x^k
1	$(-1,-1)^T$	$(-3.324e-011, 9.249e-011)^T$	$(0,0)^T$
2	$(-2,-2)^T$	$(-6.073e-011, 4.743e-011)^T$	$(0,0)^T$
3	$(-6,-3)^T$	$(-4.994e-011, 6.755e-011)^T$	$(0,0)^T$
4	$(-5,-5)^T$	$(-4.624e-011, 4.129e-011)^T$	$(0,0)^T$
5	$(1,0)^T$	$(7.735e-011, 1.134e-011)^T$	$(0,0)^T$
6	$(1,1)^T$	$(3.773e-011, 6.606e-011)^T$	$(0,0)^T$
7	$(2,3)^T$	$(1.995e-011, 5.596e-011)^T$	$(0,0)^T$
8	$(10,10)^T$	$(3.253e-011, 6.019e-011)^T$	$(0,0)^T$
9	$(-2,3)^T$	$(-6.99e-012, 6.995e-011)^T$	$(0,0)^T$
10	$(-1,1)^T$	$(-3.871e-011, 3.585e-011)^T$	$(0,0)^T$

Example 4.2. *Consider the nonsmooth equations with finitely many maximum functions given as (1.1). Similar as previously mentioned, we can obtain that*

$$F(x) = (f_1(x), f_2(x), \cdots, f_9(x))^T,$$

where $f_i(x) = x_i^2 + x_{i+1}^2, i = 1, \cdots, 8, f_9(x) = x_9^2$. *Obviously, this function has a unique solution* $x = (0,0,0,0,0,0,0,0,0)^T$. *Then we compare M1 with M3 and* ***fsolve****,respectively. The numerical results are shown in Table 4.2 and Table 4.3.*

Table 4.2

		M1	*M3*
	x^0	*N*	*N*
1	$(1,0,1,1,1,1,1,1,1)^T$	36	$\times$
2	$(1,1,0,1,1,0,1,1,1)^T$	36	$\times$
3	$(1,0,1,0,1,0,1,0,1)^T$	35	$\times$
4	$(1,1,1,1,1,1,1,1,1)^T$	36	51
5	$(1,2,3,4,5,6,7,8,9)^T$	39	54

Table 4.3

		M1	*fsolve*
	x^0	$x^k(e-11)$	$x^k(e-15)$
1	(1; 0; 1; 1; 1; 1; 1; 1; 1)	(1.9; 0.3; 2.1; 2.1; 2.1; 2.1; 2.1; 2.2; 1.9)	(0; 0; 0; 0; 0; 0; 0; 0; 0)
2	(1; 1; 0; 1; 1; 0; 1; 1; 1)	(2.2; 1.9; 0.2; 2.2; 1.9; 0.4; 2.1; 2.2; 1.9)	(0; 0; 0; 0; 0; 0; 0; 0; 0)
3	(1; 0; 1; 0; 1; 0; 1; 0; 1)	(3.8; 0.9; 3.8; 0.9; 3.8; 0.9; 3.8; 0.9; 3.9)	(0; 0; 0; 0; 0; 0; 0; 0; 0)
4	(1; 1; 1; 1; 1; 1; 1; 1; 1)	(2.1; 2.1; 2.1; 2.1; 2.1; 2.1; 2.1; 2.1; 1.9)	(0; 0; 0; 0; 0; 0; 0; 0; 0)
5	(1; 2; 3; 4; 5; 6; 7; 8; 9)	(0.3; 0.7; 0.9; 1.4; 1.7; 2.2; 2.5; 3.1; 3.0)	(0; 0; 0; 0; 0; 0; 0; 0; 0)

From Table 4.1 and Table 4.3, we know that Method 3.1 is similar to fsolve on the numerical accuracy. And from Table 4.2, we find that M1 needs less number of iterations than M3 in solving the same problem. Now, we give the following numerical examples to illustrate that Method 3.1 and Method 3.2 also adapt to semismooth equations with the Clarke generalized Jacobian, B-differential and b-differential.

Example 4.3. *([18])*

$$F(x) = \begin{pmatrix} |x_1| + (x_2-1)^2 - 1 \\ (x_1-1)^2 + |x_2| - 1 \end{pmatrix} = 0.$$

The above problem is the system of semismooth equations and it has two solutions

$$x^* = (0,0)^T, x^{**} = (1,1)^T.$$

The numerical results are shown in Table 4.4 and Table 4.5.

Table 4.4

	M3		M1	
x^0	N	x^k	N	x^k
$(0,1)^T$	-	-	14	$(1.0000, 1.0000)^T$
$(0.5,0.5)^T$	-	-	15	$(1.0000, 1.0000)^T$
$(0.5,2)^T$	33	$(1.0000, 1.0000)^T$	14	$(1.0000, 1.0000)^T$
$(1,0)^T$	-	-	14	$(1.0000, 1.0000)^T$
$(2,3)^T$	39	$(1.0000, 1.0000)^T$	24	$(1.0000, 1.0000)^T$
$(10,10)^T$	38	$(1.0000, 1.0000)^T$	17	$(1.0000, 1.0000)^T$
$(0,-1)^T$	33	$(-1.086e-011, -0.93e-011)^T$	18	$(-5.532e-012, -5.533e-012)^T$
$(0,-10)^T$	39	$(-5.143e-011, -1.748e-011)^T$	20	$(1.0000, 1.0000)^T$
$(5,-6)^T$	37	$(1.0000, 1.0000)^T$	18	$(1.0000, 1.0000)^T$
$(-1,0)^T$	33	$(-9.334e-011, -1.086e-011)^T$	18	$(-5.533e-012, -5.532e-012)^T$
$(-10,0)^T$	39	$(-1.748e-011, -5.143e-011)^T$	20	$(1.0000, 1.0000)^T$
$(-1,-1)^T$	34	$(-5.132e-011, -5.132e-011)^T$	18	$(-1.094e-011, -1.094e-011)^T$
$(-1.5,-1.5)^T$	35	$(-3.847e-011, -3.847e-011)^T$	18	$(-1.981e-011, -1.981e-011)^T$
$(-5,-5)^T$	37	$(-3.656e-011, -3.656e-011)^T$	20	$(-9.158e-012, -9.158e-012)^T$
$(-10,-10)^T$	38	$(-4.373e-011, -4.373e-011)^T$	21	$(-8.935e-012, -8.935e-012)^T$

Table 4.5

	M3		M2	
x^0	N	x^k	N	x^k
$(0,1)^T$	-	-	35	$(1.0000, 1.0000)^T$
$(0.5,0.5)^T$	-	-	34	$(1.0000, 1.0000)^T$
$(0.5,2)^T$	33	$(1.0000, 1.0000)^T$	35	$(1.0000, 1.0000)^T$
$(1,0)^T$	-	-	35	$(1.0000, 1.0000)^T$
$(2,3)^T$	39	$(1.0000, 1.0000)^T$	37	$(1.0000, 1.0000)^T$
$(10,10)^T$	38	$(1.0000, 1.0000)^T$	40	$(1.0000, 1.0000)^T$
$(0,-1)^T$	33	$(-1.086e-011, -0.93e-011)^T$	27	$(4.799e-011, 2.966e-011)^T$
$(0,-10)^T$	39	$(-5.143e-011, -1.748e-011)^T$	40	$(1.0000, 1.0000)^T$
$(5,-6)^T$	37	$(1.0000, 1.0000)^T$	39	$(1.0000, 1.0000)^T$
$(-1,0)^T$	33	$(-9.334e-011, -1.086e-011)^T$	27	$(-2.966e-011, 4.799e-011)^T$
$(-10,0)^T$	39	$(-1.748e-011, -5.143e-011)^T$	40	$(1.0000, 1.0000)^T$
$(-1,-1)^T$	34	$(-5.132e-011, -5.132e-011)^T$	18	$(-4.628e-012, -4.628e-012)^T$
$(-1.5,-1.5)^T$	35	$(-3.847e-011, -3.847e-011)^T$	18	$(-9.293e-012, -9.293e-012)^T$
$(-5,-5)^T$	37	$(-3.656e-011, -3.656e-011)^T$	20	$(-5.944e-012, -5.944e-012)^T$
$(-10,-10)^T$	38	$(-4.373e-011, -4.373e-011)^T$	21	$(-7.788e-012, -7.788e-012)^T$

Example 4.4. *([19])*

$$f_1(x) = (x_2 - x_1)ln[(x_2 - x_1)^2 + 1] + x_2 - x_1,$$

$$f_2(x) = \begin{cases} -exp(-x_1 - x_2) + 1, & if\ x_2 \geq 0; \\ \frac{1-exp(-x_1)}{1-x_2}, & if\ x_2 \leq 0. \end{cases}$$

where the function $F : R^2 \to R^2$ with component functions. And it has a unique solution $x^* = (0,0)^T$. And the results are shown in Table 4.6 and Table 4.7.

Table 4.6

	M3		M1	
x^0	N	x^k	N	x^k
$(1,0)^T$	*34*	$(5.932e-011, 6.46e-012)^T$	*20*	$(1.668e-011, -1.031e-011)^T$
$(2,2)^T$	-	-	*25*	$(-4.101e-011, 1.698e-011)^T$
$(-1,1)^T$	*34*	$(-6.302e-011, 6.302e-011)^T$	*22*	$(-1.750e-011, 7.25e-012)^T$
$(-0.5,-0.5)^T$	*33*	$(-6.338e-011, -6.338e-011)^T$	*19*	$(-1.923e-011, 7.96e-012)^T$
$(-1,-1)^T$	*34*	$(-5.493e-011, -5.493e-011)^T$	*20*	$(-1.407e-011, 5.83e-012)^T$
$(-2,-2)^T$	*35*	$(-5.116e-011, -5.116e-011)^T$	*20*	$(-3.995e-011, 1.655e-011)^T$
$(-2,-3)^T$	*35*	$(-5.331e-011, -7.974e-011)^T$	*20*	$(1.031e-011, -6.37e-012)^T$
$(-3,-3)^T$	*36*	$(-4.458e-011, -4.458e-011)^T$	*21*	$(-3.145e-011, 1.303e-011)^T$
$(5,-6)^T$	-	-	*24*	$(-1.305e-011, 5.40e-012)^T$
$(-5,-6)^T$	*37*	$(-6.517e-011, -7.177e-011)^T$	*23*	$(-2.031e-011, 8.41e-012)^T$

Table 4.7

	M3		M2	
x^0	N	x^k	N	x^k
$(1,0)^T$	*34*	$(5.932e-011, 6.46e-012)^T$	*38*	$(-4.877e-011, -7.892e-011)^T$
$(2,2)^T$	-	-	*31*	$(2.465e-011, 5.952e-011)^T$
$(-1,1)^T$	*34*	$(-6.302e-011, 6.302e-011)^T$	*31*	$(2.315e-011, 5.952e-011)^T$
$(-0.5,-0.5)^T$	*33*	$(-6.338e-011, -6.338e-011)^T$	*42*	$(-5.065e-011, -8.196e-011)^T$
$(-1,-1)^T$	*34*	$(-5.493e-011, -5.493e-011)^T$	*43*	$(-6.47e-011, -1.047e-010)^T$
$(-2,-2)^T$	*35*	$(-5.116e-011, -5.116e-011)^T$	*45*	$(-5.572e-011, -9.015e-011)^T$
$(-2,-3)^T$	*35*	$(-5.331e-011, -7.974e-011)^T$	*45*	$(-7.06e-011, -1.142e-010)^T$
$(-3,-3)^T$	*36*	$(-4.458e-011, -4.458e-011)^T$	*46*	$(-6.41e-011, -1.037e-010)^T$
$(5,-6)^T$	-	-	*48*	$(-6.18e-011, -1.001e-010)^T$
$(-5,-6)^T$	*37*	$(-6.517e-011, -7.177e-011)^T$	*48*	$(-6.68e-011, -1.080e-010)^T$

By Table 4.1 and Table 4.3, we obtain that Method 3.1 is efficient and it is similar to **fsolve** in the final solution, and Method 3.2 is invalid in Example 4.1 and Example 4.2. And Table 4.2, 4.4 and 4.6 indicate that M1 is promising since M1 need less number of iterations. From Table 4.5 and 4.7, we find that the result of M2 is similar to M3. But both M1 and M2 are promising. Especially when dealing with the problem of nonsingular matrix, our methods are more efficient than the midpoint Newton method given in [15].

5 Conclusion

Newton-type method is one of the most important tool for solving the nonsmooth equations with finitely many maximum functions, which is widely used in solving many economics, engineering and optimization problems. In this paper, we present two Newton-type methods with ∂_*-differential for solving the nonsmooth equations with finitely many maximum functions. And we prove the superlinear convergence of the given methods. Finally, we also show that the methods we propose are valid in semismooth equations with ∂_B-differential by the numerical experiments. As for the further work, we can also consider the global convergence of some methods to solve the problem of the nonsmooth equations with finitely many maximum functions.

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