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Point Rupture Solutions of Singular Elliptic Equations in N-D

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Abstract We consider the elliptic equation

$$\Delta u = f(u)$$

in a region $\Omega \subset \mathbb{R}^N,\ N \geq 3$, where f is a positive continuous function satisfying

$$\lim_{u \to 0^+} f(u) = \infty.$$

Motivated by the thin film equations, a solution u is said to be a point rupture solution if for some $p \in \Omega$, $u(p) = 0$ and $u(p) > 0$ in $\Omega \setminus \{p\}$. Solving the associated ordinary differential equations confirm our main results of sufficient conditions on f for the existence uniquness of radial point rupture solution and its asymptotic behavior. Furthermore, we can prove that our results can be applied to the point rupture solutions for a class of quasi-linear elliptic equations of the form

$$\operatorname{div}(a(u)\Delta u) = \frac{a'(u)}{2}|\nabla u|^2 + f(u)$$

Keywords: Thin film; point rupture; radial solution; singular equation; quasi-linear elliptic equation

1 Introduction

Let Ω be a region in $\mathbb{R}^N,\ N \geq 3$ and f be a positive continuous function defined on $(0, \infty)$ such that it can be written as a product of two positive, continuous functions f_1 and f_2 such that f_1 is uniformly bounded and f_2 is decreasing near zero having single zero at some positive t_0 and we require f to satisfy the usual growth condition,

$$\lim_{v \to 0^+} f(v) = \infty \tag{1.1}$$

We are interested in the elliptic equation

$$\Delta u = f(u) \text{ in } \Omega \tag{1.2}$$

This equation was studied by many authors and its rupture solution was derived in [13] when dealing with the zero set of sobolev functions having negative power of integrability. The same equation was also investigated for its rupture solution in [15] for the case $f(u) = u^{-\alpha} - 1$, for $\alpha > 1$, and in this case there are many applications to the Van der Waals force driven thin films. The equation was also used in [14], when f satisfies a special integrability condition, and in [11] when the space dimension is 3 and above.The quasi-linear equation (1.4) which can be transformed to (1.2) was analyzed for rupture solutions in [6] where they proved existence of solutions called explosive solutions.

In this we consider Ω to be a bounded smooth region in $\mathbb{R}^N$ where $N \geq 3$ and we assume that f is a positive, continuous function that can be written as a product of two positive, continuous functions f_1 and f_2 such that f_1 is uniformly bounded and f_2 is decreasing near zero having single zero at some positive t_0 and we require f to satisfy the usual growth condition. Clearly, f_2 inherits all the properties of f near zero, more precisely, it will be decreasing, positive, continuous and the same limit as f near zero. Of course our main interest comes from the general N dimensional elliptic equation

$$u_t = -\nabla \cdot (u^m \nabla u) - \nabla \cdot (u^n \nabla \Delta u). \tag{1.3}$$

The second term on the right which is the fourth-order term of the equation reflects surface tension effects, and the second-order term may reflect van der Waals interactions, gravity, the geometry of the

solid substrate or thermocapillary effects. This class of model equation is related to many physical systems involving fluid interfaces.
When $n = 1, m = 1$, it describes a thin jet in a Hele-Shaw cell [1], [5], [8], [9]; when $n = m = 3$ it describes fluid droplets hanging from a ceiling [10]; when $n = 0$ and $m = 1$, it describes solidification of a hyper-cooled melt [3], [4]; and when $n = 3, m = -1$, it models van der Waals force driven thin film [7], [12], [18], [19], [20], when the space dimension is one R. Laugesen and M. Pugh [16] studied rigorously, in a general setting, positive periodic steady states and touchdown steady states solution. F. Bernis and A. Friedman in [2] established the existence of weak solutions and showed that the support of the thin film will expand with time. Equation (1.3) models the dynamic of thin films equation, using the pressure as defined earlier with Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. The assumptions of the wetting and non-wetting of surfaces and steady states solutions lead to the semi-linear elliptic equation (1.2) . The main result guarantees the existence of a weak radial point rupture solution then this result obtained for the equation (1.2) is exploited to prove the existence of weak point rupture solutions for the quasi-linear elliptic equations of the form

$$\operatorname{div}\left(a\left(u\right)\nabla u\right) = \frac{a'\left(u\right)}{2}\left|\nabla u\right|^2 + f\left(u\right). \tag{1.4}$$

where for some $\sigma^* > 0$ $a \in C^1\left[0, \sigma^*\right]$ and $f \in C\left(0, \sigma^*\right]$ are positive functions of a real variable. Therefore, We are dealing with a semi-linear elliptic equation in $\mathbb{R}^N$, and as in the preceding, in $\mathbb{R}^N$ for now , let Ω be a smooth region in $\mathbb{R}^N$ with $N \geq 3$ and f be a positive continuous function defined on $(0, \infty)$ satisfying the growth condition (1.1). Moreover we assume the function f to be the product of two functions as mentioned above. Here also, a solution to (1.2) is said to be an N dimensional point rupture solution if for some $p \in \Omega \subset \mathbb{R}^N$, $u\left(p\right) = 0$ and $u\left(x\right) > 0$ for any $x \in \Omega \backslash \left\{p\right\}$. The main purpose is to find a sufficient condition on the growth of f near the origin so that (1.2) has a radial point rupture solution in $\mathbb{R}^N$. The main difficulty of the problem is the same as in the plane, two types of singularities involved. In radial coordinates, $\Delta u = u_{rr} + \frac{1}{r}u_r$, becomes singular when $r = 0$. Such singularity is artificial if u behaves nicely and $\frac{1}{r}u_r$ becomes continuous at $r = 0$. However, for rupture solutions, u_r itself could blow up at $r = 0$. Singularity also arises when we assume (1.1) and the solution touches zero.

2 The Main Result

The following is the statement of the main theorem where we assert the existence of an N dimensional point rupture and weak solution to the semi-linear elliptic equation with its appropriate bounds.

Theorem 1. *Let* $t_1 > 0$ *be such that* f *is a continuous, positive function and can be written as* $f = f_1 f_2$ *in* $(0, t_1]$. *Assume that* f_2 *is continuous,positive and monotone decreasing, also* f_1 *is uniformly bounded, that is, there exist positive constants* A *and* B *such that,* $A \leq f_1 \leq B$. *The function* f *is supposed to satisfy the usual blow up condition near zero, that is,* $\lim_{v\to 0^+} f\left(v\right) = \infty$. *, then define:*

$$G\left(v\right) = \int_0^v \frac{1}{f_2\left(s\right)} ds. \tag{2.1}$$

Then there exists $r^* > 0$ *and a radial point rupture solution* u_0 *to* (1.2) *in* $B_{r^*}\left(0\right)$ *such that* $u_0 = u_0\left(r\right)$ *is continuous on* $[0, r^*]$,

$$u_0\left(0\right) = 0,\ u_0\left(r\right) > 0 \text{ for any } r \in \left(0, r^*\right],$$

and u_0 *is a weak solution to* (1.2) *in* $B_{r^*}\left(0\right)$. *Moreover,* u_0 *is monotone increasing and satisfies the following bounds*

$$G^{-1}\left(\frac{Ar^2}{2N}\right) \leq u_0(r) \leq \frac{BN}{A(N-2)} G^{-1}\left(\frac{Ar^2}{2N}\right)$$

3 Proof of the Main Result

Proof. In this part we assume that $f = f_1 f_2$ with f_1 uniformly bounded and f_2 is decreasing near zero. so we can fix $t_1 > 0$ so that f_2 is decreasing in $(0, t_1]$, this is possible since f_2 is assumed to be decreasing near zero, now for any $\alpha \in (0, t_1]$ define $r_1(\alpha) = \inf\{r > 0, u_\alpha(r) = t_1\}$ and note that such r_1 exists because of the oscillation of u_α. Since $f > 0$ in $(0, t_1]$ and $u_\alpha(0) = \alpha$, we have that u_α is increasing in $(0, r_1]$ from α to t_1 since we have

$$r^{N-1} u'_\alpha(r) = \int_0^r f_1 \left(u_\alpha(s)\right) f_2 \left(u_\alpha(s)\right) s^{N-1} ds.$$

Therefore, since f_2 is decreasing in $(0, r_1]$, using the uniform bound of the function f_1, that is, $A \leq f_1 \leq B$ we get,

$$r^{N-1} u'_\alpha(r) \geq A \int_0^r f_2(u_\alpha(s)) s^{N-1} ds \geq A f_2(u_\alpha(r)) \int_0^r s^{N-1} ds.$$

Hence, since we defined $G(t) = \int_0^t \frac{1}{f_2(s)} ds$ for $t < t_0$ by noting that G is increasing in $[0, t_0]$ we conclude that

$$\int_0^r \frac{u'_\alpha(s)}{f_2(u_\alpha(s))} ds \geq \int_0^r \frac{As}{N} ds, \quad \text{thus} \quad G(u_\alpha(r)) \geq \frac{Ar^2}{2N} + G(\alpha) \geq \frac{Ar^2}{2N}$$

$$u_\alpha(r) \geq G^{-1}\left(\frac{Ar^2}{2N}\right) \quad \text{for any} \quad \alpha \in (0, t_1), \quad \text{and for all } r \in [0, r_1].$$

□

Now we will prove that there exists a constant $r^* > 0$ such that $r_1(\alpha) \geq r^*$ for all small α,

Corollary 1. *There exists an $r^* > 0$ such that for any $\alpha \in \left(0, \frac{t_1}{2}\right]$*

$$r_1(\alpha) \geq r^*.$$

We may define,

$$r^* = \sqrt{\frac{2N}{A} G\left(\frac{At_1}{6B}\right)}$$

Proof.

$$u'_\alpha(r) \leq Br^{1-N} \int_0^r f_2(u_\alpha(s)) s^{N-1} ds \leq Br^{1-N} \int_0^r f_2(G^{-1}(\frac{As^2}{2N})) s^{N-1} ds \quad \text{for} \quad r \leq r_1$$

Thus integrating from 0 to r_1 we get,

$$t_1 - \alpha \leq \int_0^{r_1} Br^{1-N} \int_0^r f_2\left(G^{-1}\left(\frac{As^2}{2N}\right)\right) s^{N-1} ds dr$$

interchanging integration we get

$$t_1 - \alpha \leq B \int_0^{r_1} f_2\left(G^{-1}\left(\frac{As^2}{2N}\right)\right) s^{N-1} \int_s^{r_1} r^{1-N} dr ds$$

hence

$$t_1 - \alpha \leq \frac{B}{N-2} \int_0^{r_1} f_2\left(G^{-1}\left(\frac{As^2}{2N}\right)\right) s ds$$

Using the change of variable $z = \frac{As^2}{2N}$ we get

$$t_1 - \alpha \le \frac{BN}{A(N-2)} \int_0^{\frac{Ar_1^2}{2N}} f_2(G^{-1}(z))dz = \frac{BN}{A(N-2)} G^{-1}\left(\frac{Ar_1^2}{2N}\right)$$

So for $0 < \alpha < \frac{t_1}{2}$ we get,

$$\frac{Ar_1^2}{2N} \ge G\left(\frac{A(N-2)t_1}{B2N}\right) \ge G\left(\frac{At_1}{6B}\right)$$

Therefore $r_1(\alpha)$ is uniformly bounded from below for any $\alpha > 0$ small enough.

Take r^* to be $r^* = \sqrt{\frac{2N}{A} G\left(\frac{At_1}{6B}\right)}$.

□

Now, observe that for any $\alpha < \delta < r \le r^*$ we have

$$G^{-1}(\frac{Ar^2}{2N}) \le u_\alpha(r) \le t_1 \le \frac{BN}{A(N-2)} G^{-1}\left(\frac{Ar_1^2}{2N}\right) + \alpha. \tag{3.1}$$

Using standard elliptic theory and diagonal process we can construct a subsequence u_α that converges locally and uniformly to the rupture solution u_0 that satisfies $\Delta u_0 = f(u_0)$.

Proposition 1. *There exists a sequence* $\{\alpha_k\}_{k=1}^{\infty} \subset \left(0, \frac{t_1}{2}\right]$ *satisfying*

$$\lim_{k\to\infty} \alpha_k = 0,$$

such that $u_{\alpha_k} \to u_0$ *uniformly in* $\overline{B_{r^*}(0)}$ *as* $k \to \infty$, *for some function*

$$u_0 \in C^0\left(\overline{B_{r^*}(0)}\right) \cap C^2\left(\overline{B_{r^*}(0)} \setminus \{0\}\right).$$

Moreover, u_0 *is a classical solution to* (1.2) *in* $B_{r^*}(0) \setminus \{0\}$ *and*

$$G^{-1}(\frac{Ar^2}{2N}) \le u_0(r) \le \frac{BN}{A(N-2)} G^{-1}\left(\frac{Ar^2}{2N}\right)$$

Proof. For any $\varepsilon > 0$, u_α, $\alpha \in \left(0, \frac{t_1}{2}\right]$ is a family of uniformly bounded classical solutions to

$$\Delta u = f(u) \text{ in } \overline{B_{r^*}(0)} \setminus B_\varepsilon(0),$$

hence by a diagonal argument, there exists a sequence $\{\alpha_k\}_{k=1}^{\infty} \subset \left(0, \frac{t_1}{2}\right]$ satisfying $\lim_{k\to\infty} \alpha_k = 0$, such that $u_{\alpha_k} \to u_0$ locally uniformly in $\overline{B_{r^*}(0)} \setminus \{0\}$ as $k \to \infty$. Now (3.1) implies

$$G^{-1}(\frac{Ar^2}{2N}) \le u_0(r) \le \frac{BN}{A(N-2)} G^{-1}\left(\frac{Ar^2}{2N}\right)$$

Since

$$\lim_{r\to 0} \frac{BN}{A(N-2)} G^{-1}\left(\frac{Ar^2}{2N}\right) = 0,$$

it is not difficult to see, from the bounds of u_α and u_0, that $u_{\alpha_k} \to u_0$ uniformly in $\overline{B_{r^*}(0)}$ as $k \to \infty$. □

Remark 1. *The above limit should be independent of the choice of the sequence* $\{\alpha_k\}_{k=1}^{\infty}$. *Actually, we expect that* $u_\alpha \to u_0$ *uniformly on* $[0, r^*]$ *as* $\alpha \to 0$. *This is an open question.*

In order to show that u_0 is a weak solution. The following lemma will be very useful.

Lemma 1. *Let v_0 be the rupture radial solution in $\mathbb{R}^n$ of,*

$$\Delta v = h(v) \qquad then \quad \lim_{r\to 0^+} r^{n-1} v_0'(r) = 0.$$

Proof. We have the followings,

$$\Delta v_0 = h(v_0), \quad \text{since radial in } \mathbb{R}^n, \quad v_0'' + \frac{n-1}{r} v_0' = h(v_0)$$

$$r^{n-1} v_0'' + (n-1) r^{n-2} v_0'(r) = r^{n-1} h(v_0) \quad \text{or} \quad (r^{n-1} v_0')' = r^{n-1} h(v_0) > 0$$

Hence, we see that $r^{n-1} v_0'(r)$ is monotone increasing in $(0, r^*)$. Since $r^{n-1} v_0' \geq 0$ in $(0, r^*)$

$$\beta = \lim_{r\to 0^+} r^{n-1} v_0'(r) \geq 0.$$

is well defined. Assuming that $\beta > 0$, there exists $\delta > 0$ such that for $r \in (0, \delta]$ we have,

$$r^{n-1} v_0'(r) \geq \frac{\beta}{2} \quad \text{thus we get } v_0(r) \leq v_0(\delta) - \frac{\beta \delta^{2-n}}{2-n} + \frac{\beta}{2(2-n) r^{n-2}}, \text{ which is a contradiction.}$$

□

Now, in $\mathbb{R}^n$ the rupture solution for, $\Delta v = h(v)$ is a weak solution in $\Omega = B_{r^*}(0)$.

Proposition 2. $h(v_0) \in L^1(\Omega)$ *and v_0 is a weak solution to,* $\Delta v = h(v)$.

Proof. For any test function $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega v_0 \Delta\varphi dx = \lim_{\varepsilon\to 0^+} \int_{\Omega\setminus\overline{B_\varepsilon(0)}} v_0 \Delta\varphi dx \\
&= \lim_{\varepsilon\to 0^+} \left(\int_{\Omega\setminus\overline{B_\varepsilon(0)}} \Delta v_0 \varphi dx - \int_{\partial B_\varepsilon(0)} \left(v_0 \frac{\partial\varphi}{\partial n} - \varphi \frac{\partial v_0}{\partial n} \right) ds_x \right) \\
&= \lim_{\varepsilon\to 0^+} \left(\int_{\Omega\setminus\overline{B_\varepsilon(0)}} h(v_0) \varphi dx - \int_{\partial B_\varepsilon(0)} v_0 \frac{\partial\varphi}{\partial n} ds_x + \int_{\partial B_\varepsilon(0)} \varphi \frac{\partial v_0}{\partial n} ds_x \right).
\end{aligned}$$

Now for any $\varepsilon \in (0, r^*)$, since $v_0(\varepsilon) \leq v_0(r^*) \leq t_1$, we have

$$\begin{aligned}
\left| \int_{\partial B_\varepsilon(0)} v_0 \frac{\partial\varphi}{\partial n} ds_x \right| &\leq v_0(\varepsilon) \|\nabla\varphi\|_{L^\infty(\Omega)} |\partial B_\varepsilon(0)| \\
&\leq n\alpha(n) \varepsilon^{n-1} v_0(\varepsilon) \|\nabla\varphi\|_{L^\infty(B_{r^*}(0))} \to 0
\end{aligned}$$

as $\varepsilon \to 0^+$. Here $n\alpha(n)$ denotes the surface area of the unit sphere. On the other hand, the previous lemma implies that

$$\left| \int_{\partial B_\varepsilon(0)} \varphi \frac{\partial v_0}{\partial n} ds_x \right| \leq n\alpha(n) \varepsilon^{n-1} v_0'(\varepsilon) \|\varphi\|_{L^\infty(B_{r^*}(0))} \to 0$$

as $\varepsilon \to 0^+$. Hence, we have for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_{B_{r^*}(0)} v_0 \Delta\varphi dx = \lim_{\varepsilon\to 0^+} \int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} h(v_0)\varphi dx.$$

Choosing φ such that $\varphi \equiv 1$ near the origin, the above limit implies that $h(v_0)$ is integrable near the origin. Since $h(v_0)$ is a positive continuous function in $B_{r^*}(0)\setminus\{0\}$, we conclude $h(v_0) \in L^1(B_{r^*}(0))$. So we have for any test function $\varphi \in C_c^\infty(B_{r^*}(0))$

$$\int_{B_{r^*}(0)} v_0 \Delta\varphi dx = \lim_{\varepsilon\to 0^+} \int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} h(v_0)\varphi dx = \int_{B_{r^*}(0)} h(v_0)\varphi dx,$$

i.e., v_0 is a weak solution in $B_{r^*}(0) \subseteq \mathbb{R}^n$. □

4 Application to Quasi-Linear Equations

Our previous result can easily be applied in $\mathbb{R}^n$ to the point rupture solutions of the quasi-linear elliptic equations of the form

$$\operatorname{div}(a(u)\Delta u) = \frac{a'(u)}{2}|\nabla u|^2 + f(u) \tag{4.1}$$

Now let us state the theorem for an n dimensional quasi-linear elliptic equation.

Theorem 2. *Assume that for some $\sigma^* > 0$, $a \in C^1[0,\sigma^*]$ $f \in C(0,\sigma^*]$ are positive functions. Let f_1, f_2 be continuous, f_2 is monotone decreasing f_1 is uniformly bounded, such that*

$$\lim_{v\to 0^+} f_2(v) = \infty, \quad \text{and} \quad f = f_1 f_2$$

Let

$$G(v) = \int_0^v \frac{1}{f_2(s)} ds. \tag{4.2}$$

Then there exists $r^* > 0$ and a radial point rupture solution u_0 to (1.4) in $B_{r^*}(0)$ such that $u_0 = u_0(r)$ is continuous on $[0, r^*]$,

$$u_0(0) = 0,\ u_0(r) > 0 \text{ for any } r \in (0, r^*].$$

Moreover there exists a function g, and constants K_1, K_2 such that u_0 is a weak solution with the following estimates

$$g\left(G^{-1}\left(\frac{Ar^2}{2N}\right)\right) \le u_0(r) \le g\left(\frac{K_2 N}{K_1(N-2)} G^{-1}\left(\frac{K_1 r^2}{2N}\right)\right)$$

Proof. We consider the quasi-linear equation (1.4) in a region $\Omega \subset \mathbb{R}^n$ where for some $\delta^* > 0$, $a \in C^1[0,\delta^*]$ and $f \in C(0,\delta^*]$ are positive functions of a real variable. A solution to (1.4) is said to be a point rupture solution if for some $p \in \Omega$, $u(p) = 0$ and $u(x) > 0$ for any $x \in \Omega\setminus\{p\}$.

Let g be a solution to the Cauchy problem

$$g' = \frac{1}{\sqrt{a\left(g\right)}},\ g\left(0\right) = 0,$$

and let v be a solution to the elliptic problem

$$\triangle v = h\left(v\right) \tag{4.3}$$

where

$$h\left(v\right) = \frac{f\left(g\left(v\right)\right)}{\sqrt{a\left(g\left(v\right)\right)}}.$$

Define the auxiliary function u as $u = g \circ v$ from $\mathbb{R}^N$ to $\mathbb{R}$ which is well defined since v is a solution of (4.3) and h is continuous, then

$$u = g\left(v\right) \quad \text{thus} \quad h(v) = \frac{f(g(v))}{\sqrt{a(g(v))}} = \frac{f(u)}{\sqrt{a(u)}}.$$

Therefore it is clear that,

$$\nabla u = g'(v)\nabla v$$

hence,

$$\nabla u = \frac{1}{\sqrt{a(g(v))}}\nabla v,$$

thus

$$\nabla u = \frac{1}{\sqrt{a(u)}}\nabla v$$

Therefore we have

$$\nabla v = \sqrt{a\left(u\right)}\nabla u,$$

On the other hand

$$\nabla\sqrt{a(u)} = \frac{1}{2}(a(u))^{\frac{-1}{2}}a'(u)\nabla u = \frac{a'(u)}{2\sqrt{a(u)}}\nabla u,$$

Now combining all of the above leads to

$$\Delta v = \sqrt{a\left(u\right)}\Delta u + \frac{1}{2}\frac{1}{\sqrt{a\left(u\right)}}a'\left(u\right)\left|\nabla u\right|^2.$$

Hence (4.3) implies

$$\sqrt{a\left(u\right)}\Delta u + \frac{1}{2}\frac{1}{\sqrt{a\left(u\right)}}a'\left(u\right)\left|\nabla u\right|^2 = \frac{f\left(u\right)}{\sqrt{a\left(u\right)}}$$

that is,

$$a\left(u\right)\Delta u + \frac{1}{2}a'\left(u\right)\left|\nabla u\right|^2 = f\left(u\right),$$

and now by adding to both sides the quantity,

$$\frac{1}{2}a'\left(u\right)\left|\nabla u\right|^2$$

then we end up with,

$$a\left(u\right)\Delta u + a'\left(u\right)\left|\nabla u\right|^2 = \frac{1}{2}a'\left(u\right)\left|\nabla u\right|^2 + f\left(u\right),$$

which is equivalent to (1.4). Hence, (1.4) possesses a point rupture solution if and only if (4.3) has a point rupture solution.

Now h can be written as a product of two function satisfying conditions of the previous theorem

$$h(v) = \frac{f(g(v))}{\sqrt{a(g(v))}} = \frac{f_1(g(v))}{\sqrt{a(g(v))}} f_2(g(v)) = h_1(v)h_2(v)$$

where, $h_1 = \frac{f_1(g(v))}{\sqrt{a(g(v))}}$ is uniformly bounded, say $K_1 \leq K_2$ since both f_1 and g are uniformly bounded. The function h_2 defined by $h_2(v) = f_2(g(v))$ is decreasing since $g' > 0$. Therefore the technical assumptions on f_1 and f_2 imply that the function h satisfies the conditions of theorem 5.1. Hence, there exists a weak radial rupture solution v_0 which is equivalent to say that $u_0 = g(v_0)$ is a radial rupture solution for the quasi-linear equation with the bounds,

$$g\left(G^{-1}\left(\frac{Ar^2}{2N}\right)\right) \leq u_0(r) \leq g\left(\frac{K_2 N}{K_1(N-2)} G^{-1}\left(\frac{K_1 r^2}{2N}\right)\right)$$

□

Now we will prove that in $\mathbb{R}^n$ the rupture solution for the quasi-linear equation is a weak solution in $\Omega = B_{r^*}(0)$.

Proposition 3. *$f(u_0) \in L^1(\Omega)$ and u_0 is a weak solution for the quasi-linear equation in $\Omega = B_{r^*}(0)$*

Proof. Assume that u_0 is a rupture solution for the quasi-linear equation, that is u_0 satisfies,

$$\operatorname{div}(a(u)\nabla u) = \frac{a'(u)}{2}|\nabla u|^2 + f(u)$$

where $a \in C^1$, $f \in C^0$ are positive functions and $u_0(0) = 0$.

We need to show that for any test function $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \operatorname{div}(a(u_0)\nabla u_0)\varphi dx = \int_\Omega \left(\frac{a'(u_0)}{2}|\nabla u_0|^2 + f(u_0)\right)\varphi dx$$

$$\int_\Omega \left(a(u_0)\Delta u_0 + \frac{1}{2}a'(u_0)|\nabla u_0|^2 - f(u_0)\right)\varphi dx = 0 \qquad \text{that is,}$$

$$\int_\Omega \sqrt{a(u_0)}\left[\sqrt{a(u_0)}\Delta u_0 + \frac{1}{2\sqrt{a(u_0)}}a'(u_0)|\nabla u_0|^2 - \frac{f(u_0)}{\sqrt{a(u_0)}}\right]\varphi dx = 0 \qquad \text{thus,}$$

$$\int_\Omega \left[\sqrt{a(u_0)}\Delta u_0 + \frac{1}{2\sqrt{a(u_0)}}a'(u_0)|\nabla u_0|^2 - \frac{f(u_0)}{\sqrt{a(u_0)}}\right]\sqrt{a(u_0)}\varphi dx = 0 \qquad \text{hence,}$$

$$\int_\Omega [\Delta v_0 - h(v_0)]\sqrt{a(g(v_0))}\varphi dx = 0 \qquad \text{therefore we need to show}$$

$$\int_\Omega \Delta v_0 \sqrt{a(g(v_0))}\varphi dx = \int_\Omega h(v_0)\sqrt{a(g(v_0))}\varphi dx$$

Define $\psi = \sqrt{a(g(v_0))}\,\varphi$ and then let us start computing,

$$\int_{\Omega} v_0 \Delta\psi dx = \lim_{\varepsilon\to 0^+} \int_{\Omega\setminus\overline{B_\varepsilon(0)}} v_0 \Delta\psi dx = \lim_{\varepsilon\to 0^+} \left[\int_{\Omega\setminus\overline{B_\varepsilon(0)}} \psi \Delta v_0 dx - \int_{\partial B_\varepsilon(0)} \left(v_0 \frac{\partial\psi}{\partial n} - \psi \frac{\partial v_0}{\partial n} \right) ds_x \right]$$
$$= \lim_{\varepsilon\to 0^+} \left[\int_{\Omega\setminus\overline{B_\varepsilon(0)}} h(v_0)\psi dx - \int_{\partial B_\varepsilon(0)} \left(v_0 \frac{\partial\psi}{\partial n} - \psi \frac{\partial v_0}{\partial n} \right) ds_x \right]$$

Now let's justify that the boundary terms are zeros, indeed for any $\varepsilon \in (0, r^*)$ since $v_0(\varepsilon) \le v_0(r^*)$, we have

$$\left| \int_{\partial B_\varepsilon(0)} \psi \frac{\partial v_0}{\partial n} ds_x \right| \le M n\alpha(n) \varepsilon^{n-1} v_0'(\varepsilon) \|\varphi\|_{L^\infty(B_{r^*}(0))} \to 0 \text{ as } \varepsilon \to 0$$

Here M is a constant and we use the lemma. On the other hand the second boundary term can be controlled as follows,

$$\left| \int_{\partial B_\varepsilon(0)} v_0 \frac{\partial\psi}{\partial n} ds_x \right| \le \left[\left(M \|\nabla\varphi\|_{L^\infty(B_{r^*}(0))} \right) + \left(N \|\varphi\|_{L^\infty(B_{r^*}(0))} \right) v_0'(\varepsilon) \right] n\alpha(n)\varepsilon^{n-1}$$

Where $N = max_{B_{r^*}(0)} \frac{a'(g(v_0))g'(v_0)}{2\sqrt{a(g(v_0))}}$
Therefore we have that,

$$\left| \int_{\partial B_\varepsilon(0)} v_0 \frac{\partial\psi}{\partial n} ds_x \right| \to 0^+ \text{ as } \varepsilon \to 0^+$$

Hence, $f(u_0) \in L^1(\Omega)$ and u_0 is a weak solution for the quasi-linear equation in $\Omega = B_{r^*}(0)$.

□

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