On the Hermite Positive Definite Solution of the Matrix Equation $X + A^* \left(I_m \otimes X - C\right)^{-t} A = Q$

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Abstract. The nonlinear matrix equation $X + A^* \left(I_m \otimes X - C\right)^{-t} A = Q$ is considered in this paper. A sufficient and necessary condition for Hermite positive definite (HPD) solutions of this equation is presented by using properties of the Kronecker product. Two iterative methods for solving the equation are constructed by making use of the principle of the bounded sequence. The feasibility and effectiveness of the iterative methods are verified by numerical example.

Keywords: nonlinear matrix equation; Hermite positive definite solution; fixed point iteration; inverse-free iteration.

1 Introduction

In this paper, we consider the nonlinear matrix equation

$$X + A^* \left(I_m \otimes X - C\right)^{-t} A = Q \tag{1}$$

where Q is a $n \times n$ positive definite matrix, C is a mn positive semi-definite matrix, A is a $mn \times n$ complex matrix, t is arbitrary positive real number and $I_m \otimes C > 0$. We seek the HPD solution of the equation (1) in the matrix set $X \in \phi\left(n\right) = \left\{I_m \otimes X > C\right\}$.

Nonlinear matrix equations with the form of (1) have many applications in control theory, ladder networks, dynamic programming, statistics, and so on. Several authors have studied the necessary and sufficient conditions of the existence of HPD solutions of similar kinds of nonlinear matrix equations. In [1], the case $t = 1$ is considered and different iterative methods for computing the HPD solutions are proposed. In addition, the author [10, 12] considered the nonlinear matrix equation $X^s + \sum_{i=1}^{m} A_i^* X^{t_i} A_i = Q,\quad s > 0, 0 < t_i \le 1$, and proposed the effective iterative algorithm.

The paper is organized as follows: In Section 2, we give lemmas that will be needed to develop this work. Then in Section 3, we propose necessary and sufficient conditions for the existence of HPD solutions of Eq.(1). In section 4, we propose fixed point iteration algorithm and inverse-free iteration algorithm for the HPD solutions of Eq.(1). Finally, some numerical examples are presented to ensure the performance and the efficiency of the algorithm.

2 Preliminaries

Lemma 2.1. *Let* X *be an arbitrary solution of the equation (1) in* $\phi\left(n\right) = \left\langle X \middle| \hat{X} > C\right\rangle$, *then* $X \le Q$.

Lemma2.2[2]. *If* $A \ge B > 0$ *(or* $A > B > 0$ *), then* $A^\partial \ge B^\partial > 0$ *(or* $A^\partial > B^\partial > 0$ *) for all* $\partial \in (0,1]$, *and* $B^\partial \ge A^\partial > 0$ *(or* $B^\partial > A^\partial > 0$ *) for all* $\partial \in [-1,0)$.

Lemma2.3[4]. *If* D *and* E *are Hermitian matrices of the same order with* $E > 0$, *then* $DED + E^{-1} \ge 2D$.

3 Necessary and Sufficient Conditions for the Solution of the Matrix Equation (1)

Eq.(1) is equivalent to

$$I_m \otimes X + \left(I_m \otimes A^*\right)\left[I_m \otimes \left(I_m \otimes X - C\right)^{-t}\right]\left(I_m \otimes A\right) = I_m \otimes Q,$$

let

$$Y = I_m \otimes X - C,\ \tilde{Q} = I_m \otimes Q - C,\ \tilde{A} = I_m \otimes A,$$

then Eq.(1) turns into the following form

$$Y + \tilde{A}^*\left(I_m \otimes Y^{-t}\right)\tilde{A} = \tilde{Q}, \tag{2}$$

Let $\tilde{A} = I_m \otimes A$ be partitioned as

$$\tilde{A} = (A_1^T, A_2^T, \cdots, A_m^T)^T,$$

where $A_i = e_i^T \otimes A,\ i = 1,2,\cdots m,$ then Eq.(2) turns into the following form

$$Y + \sum_{i=1}^{m} A_i^* Y^{-t} A_i = \tilde{Q}. \tag{3}$$

Through the transformation process, we can get that the solution of the Eq.(1) is equivalent to the solution of the Eq.(3)

Theorem 3.1. *Eq.(3) has an HPD solution if and only if* A_i *can be factored as*

$$A_i = \left(L^* L\right)^{\frac{t}{2}} N_i \tilde{Q}^{\frac{1}{2}},\ \ i = 1,2,\cdots m \tag{4}$$

where L *is a* $n \times n$ *nonlinear matrix and* $(L\tilde{Q}^{-\frac{1}{2}}, N_1, N_2, \cdots, N_m)^T$ *has orthonormal columns. In this case, Eq.(3) has an HPD solution* $Y = L^* L$ *, and all the solutions can be constructed by this way.*

Proof.
Necessity. If Eq.(3) has an HPD solution Y, then $Y > 0$. Let $Y = L^* L$ be the Cholesky factorization, where L is a nonlinear matrix. Then Eq.(3) can be rewritten as

$$\tilde{Q}^{-\frac{1}{2}} L^* L \tilde{Q}^{-\frac{1}{2}} + \sum_{i=1}^{m} \tilde{Q}^{-\frac{1}{2}} A_i^* \left(L^* L\right)^{-\frac{t}{2}} \left(L^* L\right)^{-\frac{t}{2}} A_i \tilde{Q}^{-\frac{1}{2}} = I_{mn}, \tag{5}$$

Let $N_i = \left(L^* L\right)^{-\frac{t}{2}} A_i \tilde{Q}^{-\frac{1}{2}},\ \ i = 1,2,\cdots m$, then

$$A_i = \left(L^* L\right)^{\frac{t}{2}} N_i \tilde{Q}^{\frac{1}{2}}, i = 1,2,\cdots, m,$$

then Eq.(5) turns into the following form

$$\tilde{Q}^{-\frac{1}{2}} L^* L \tilde{Q}^{-\frac{1}{2}} + \sum_{i=1}^{m} N_i^* N_i = I_{mn},$$

that is $\begin{pmatrix} L\tilde{Q}^{-\frac{1}{2}} \\ N_1 \\ N_2 \\ \vdots \\ N_m \end{pmatrix}^* \begin{pmatrix} L\tilde{Q}^{-\frac{1}{2}} \\ N_1 \\ N_2 \\ \vdots \\ N_m \end{pmatrix} = I_{mn}$, then we get that $\begin{pmatrix} L\tilde{Q}^{-\frac{1}{2}} \\ N_1 \\ N_2 \\ \vdots \\ N_m \end{pmatrix}$ has orthonormal columns.

Sufficiency. Because $A_i = \left(L^* L\right)^{\frac{t}{2}} N_i \tilde{Q}^{\frac{1}{2}}, i = 1,2,\cdots m,$ let $Y = L^* L$, then

$$
\begin{aligned}
Y+\sum_{i=1}^{m}A_i^{*}Y^{-t}A_i &= L^{*}L+\sum_{i=1}^{m}\tilde{Q}^{\frac{1}{2}}N_i^{*}N_i\tilde{Q}^{\frac{1}{2}} \\
&= \tilde{Q}^{\frac{1}{2}}\left(\tilde{Q}^{-\frac{1}{2}}L^{*}L\tilde{Q}^{-\frac{1}{2}}+\sum_{i=1}^{m}N_i^{*}N_i\right)\tilde{Q}^{\frac{1}{2}} \\
&= \tilde{Q}^{\frac{1}{2}}\begin{pmatrix} L\tilde{Q}^{-\frac{1}{2}} \\ N_1 \\ N_2 \\ \vdots \\ N_m \end{pmatrix}^{*}\begin{pmatrix} L\tilde{Q}^{-\frac{1}{2}} \\ N_1 \\ N_2 \\ \vdots \\ N_m \end{pmatrix}\tilde{Q}^{\frac{1}{2}} \\
&= \tilde{Q}
\end{aligned}
$$

hence Eq.(3) has an HPD solution $Y = L^{*}L$.

Theorem 3.2. *Eq.(1) has an HPD solution if and only if there is a* $n\times n$ *nonlinear matrix* L *and a* $n\times n$ *matrix* $\tilde{N}$ *, such that* $A=\left(L^{*}L\right)^{\frac{t}{2}}\tilde{N}$ *, and*

$$L^{*}L+I_m\otimes\left(\tilde{N}^{*}\tilde{N}\right)=\tilde{Q}, \tag{6}$$

In this case, Eq.(1) has an HPD solution $X=Q-A^{*}\left(L^{*}L\right)^{-t}A$ *.*

Proof.

Necessity. If Eq.(3) has an HPD solution Y, then by Theorem 3.1, A_i has decomposition formula (4), and

$$\begin{pmatrix} L\tilde{Q}^{-\frac{1}{2}} \\ N_1 \\ N_2 \\ \vdots \\ N_m \end{pmatrix}^{*}\begin{pmatrix} L\tilde{Q}^{-\frac{1}{2}} \\ N_1 \\ N_2 \\ \vdots \\ N_m \end{pmatrix}=I_{mn}, \tag{7}$$

Because $A_i=e_i^{T}\otimes A$, then

$$e_i^{T}\otimes A=\left(L^{*}L\right)^{\frac{t}{2}}N_i\tilde{Q}^{\frac{1}{2}},$$

L is a nonlinear matrix, then

$$\left(L^{*}L\right)^{-\frac{t}{2}}(e_i^{T}\otimes A)=e_i^{T}\otimes\left(\left(L^{*}L\right)^{-\frac{t}{2}}A\right)=N_i\tilde{Q}^{\frac{1}{2}}, \tag{8}$$

dividing the right side of Eq.(8) $N_i\tilde{Q}^{\frac{1}{2}}$ into the same $1\times m$ block form as the middle section of Eq. (8), and setting the *i*-th block as $\tilde{N}_i$, we can get

$$N_i\tilde{Q}^{\frac{1}{2}}=e_i^{T}\otimes\tilde{N}_i=e_i^{T}\otimes\left(\left(L^{*}L\right)^{-\frac{t}{2}}A\right)$$

that is $\tilde{N}_i=\left(L^{*}L\right)^{-\frac{t}{2}}A$, $i=1,2,\cdots,m$.

Let $\left(L^{*}L\right)^{-\frac{t}{2}}A=\tilde{N}$, then $A=\left(L^{*}L\right)^{\frac{t}{2}}\tilde{N}$, $N_i=\left(e_i^{T}\otimes\tilde{N}\right)\tilde{Q}^{-\frac{1}{2}}$, combining (7), then we have

$$\begin{aligned}
&\tilde{Q}^{-\frac{1}{2}}L^*L\tilde{Q}^{-\frac{1}{2}}+\sum_{i=1}^{m}\tilde{Q}^{-\frac{1}{2}}\left(e_i\otimes\tilde{N}^*\right)\left(e_i\otimes\tilde{N}\right)\overline{Q}^{-\frac{1}{2}}\\
&=\tilde{Q}^{-\frac{1}{2}}L^*L\tilde{Q}^{-\frac{1}{2}}+\tilde{Q}^{-\frac{1}{2}}\left[\sum_{i=1}^{m}\left(e_ie_i^T\otimes\tilde{N}^*\tilde{N}\right)\right]\tilde{Q}^{-\frac{1}{2}}\\
&=\tilde{Q}^{-\frac{1}{2}}\left[L^*L+\left(\sum_{i=1}^{m}e_ie_i^T\right)\otimes\left(\tilde{N}^*\tilde{N}\right)\right]\tilde{Q}^{-\frac{1}{2}}\\
&=\tilde{Q}^{-\frac{1}{2}}\left[L^*L+I_m\otimes\left(\tilde{N}^*\tilde{N}\right)\right]\tilde{Q}^{-\frac{1}{2}}\\
&=I_{mn}
\end{aligned}$$

Hence $L^*L+I_m\otimes\left(\tilde{N}^*\tilde{N}\right)=\tilde{Q}$.

Sufficiency. If $A=\left(L^*L\right)^{\frac{t}{2}}\tilde{N}$, L are nonlinear, and by (6), then

$$A_i=e_i^T\otimes A=e_i^T\otimes\left(\left(L^*L\right)^{\frac{t}{2}}\tilde{N}\right)=\left(L^*L\right)^{\frac{t}{2}}\left(e_i^T\otimes\tilde{N}\right),$$

Let $N_i=\left(e_i^T\otimes\tilde{N}\right)\tilde{Q}^{-\frac{1}{2}}$, then $A_i=\left(L^*L\right)^{\frac{t}{2}}N_i\tilde{Q}^{\frac{1}{2}}$, combining (6), it is not difficult to verify

$$\tilde{Q}^{-\frac{1}{2}}L^*L\tilde{Q}^{-\frac{1}{2}}+\sum_{i=1}^{m}N_i^*N_i=I_{mn},$$

Hence $(L\tilde{Q}^{-\frac{1}{2}},N_1,N_2,\cdots,N_m)^T$ has orthonormal columns. By Lemma 2.1, Eq.(3) has an HPD solution, thus Eq.(1) has an HPD solution, and $Y=L^*L$ is a solution of Eq.(3). When this is substituted in Eq.(1), we get

$$X=Q-A^*Y^{-t}A=Q-A^*\left(L^*L\right)^{-t}A.$$

4 An Iterative Method for Solving the Equation (1)

For convenience, $I_m\otimes X$ is expressed as $\hat{X}$ in the following.

Algorithm 4.1.

$$\begin{cases}X_0=Q,\\ X_{n+1}=Q-A^*\left(\hat{X}_n-C\right)^{-t}A,\ n=0,1,2\cdots\end{cases}$$

Theorem 4.1. *Set $X\in\phi\left(n\right)$ as arbitrary HPD solution of Eq. (1), and $0<t\leq 1$, then the sequence $\left\{X_n\right\}$ generated by Algorithm 4.1 converges to the maximal solution X_L of the Eq. (1) in $\phi\left(n\right)$.*

Proof. For arbitrary HPD solution X of Eq. (1), suppose that $X_k\geq X$, then $\hat{X}_k\geq\hat{X}$, by Lemma 2.2, we have

$$\left(\hat{X}_k-C\right)^{-t}\leq\left(\hat{X}-C\right)^{-t}\Rightarrow Q-A^*\left(\hat{X}_k-C\right)^{-t}A\geq Q-A^*\left(\hat{X}-C\right)^{-t}A,$$

i.e., $X_{k+1}\geq X$, thus we have $X_n\geq X,\ n=1,2,\cdots$.

Next we shall prove that $\left\{X_n\right\}$ is monotonically decreasing.

$$X_0=Q,\ X_1=Q-A^*\left(\hat{X}_0-C\right)^{-t}A<Q=X_0,\ \text{i.e.,}\ X_1<X_0$$

Suppose that $X_k<X_{k-1}$, then $\hat{X}_k<\hat{X}_{k-1}$,

$$X_{k+1}=Q-A^*\left(\hat{X}_k-C\right)^{-t}A,$$

$$X_k = Q - A^*\left(\hat{X}_{k-1} - C\right)^{-t} A \,,$$

Since $\hat{X}_k < \hat{X}_{k-1}$, then we obtain

$$(\hat{X}_k - C)^{-t} > (\hat{X}_{k-1} - C)^{-t} \Rightarrow Q - A^*(\hat{X}_k - C)^{-t} A < Q - A^*(\hat{X}_{k-1} - C)^{-t} A \,,$$

i.e., $X_{k+1} < X_k$,

Thus by induction, we get that $\{X_n\}$ is monotonically decreasing.

Consequently, $\{X_n\}$ is monotonically decreasing and has lower bound, thus we have $X_n \to X_L$.

We give the inversion-free iteration algorithm for solving the Eq.(1). Without loss of generality, we assume that $Q = I_n$ in the Eq.(1), then $\hat{Q} > C$ translates into $\hat{I}_n > C$.

Here, the Eq.(1) becomes $X + A^*\left(I_m \otimes X - C\right)^{-t} A = I_n$.

Algorithm 4.2.

$$\begin{cases} X_0 = I_n, Y_0 = \left(\hat{I}_n - C\right)^{-1}, \\ X_{k+1} = I_n - A^* Y_k^t A, \\ Y_{k+1} = Y_k\left[2\hat{I}_n - \left(\hat{X}_k - C\right)Y_k\right], k = 0,1,2,\cdots \end{cases}$$

Theorem 4.2. *Suppose that the sequence* $X_n, Y_n, n = 0,1,2\cdots$ *generated by Algorithm 4.2,* $X \in \phi(n)$ *is arbitrary HPD solution of Eq. (1), and* $0 < t \le 1$, *then the sequence* $\{X_n\}$ *generated by Algorithm 4.2 converges to the maximal solution* X_L *of the Eq. (1) in* $\phi(n)$.

Proof. Since X_L is an HPD solution of Eq.(1), then $X_L \le I_n$, and then $\hat{X}_L \le \hat{I}_n$, thus we have $X_0 = I_n \ge X_L$, $Y_0 = \left(\hat{I}_n - C\right)^{-1} \le \left(\hat{X}_L - C\right)^{-1}$.

According to Lemma 2.2 and Lemma 2.3, we obtain

$$\begin{aligned} Y_1 &= Y_0\left[2\hat{I}_n - \left(\hat{X}_0 - C\right)Y_0\right] \\ &= 2Y_0 - Y_0\left(\hat{X}_0 - C\right)Y_0 \\ &\le \left(\hat{X}_0 - C\right)^{-1} \le \left(\hat{X}_L - C\right)^{-1}. \\ Y_1 - Y_0 &= Y_0 - Y_0\left(\hat{X}_0 - C\right)Y_0 \\ &= Y_0\left(Y_0^{-1} - \left(\hat{X}_0 - C\right)\right)Y_0 = 0. \end{aligned}$$

From Lemma 2.2 and $Y_0 \le \left(\hat{X}_L - C\right)^{-1}$, we obtain

$$X_1 = I_n - A^* Y_0^t A \ge I_n - A^*\left(\hat{X}_L - C\right)^{-t} A = X_L \,,$$

and $X_1 - X_0 = -A^* Y_0^t A < 0$,

thus $X_0 > X_1 \ge X_L$ and $Y_0 = Y_1 \le \left(\hat{X}_L - C\right)^{-1}$.

Suppose that $X_{k-1} > X_k \ge X_L$ and $Y_{k-1} < Y_k \le \left(\hat{X}_L - C\right)^{-1}$,

We shall prove the inequalities $X_k > X_{k+1} \ge X_L$ and $Y_k < Y_{k+1} \le \left(\hat{X}_L - C\right)^{-1}$.

By Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} Y_{k+1} &= Y_k\left[2\hat{I}_n - \left(\hat{X}_k - C\right)Y_k\right] \\ &= 2Y_k - Y_k\left(\hat{X}_k - C\right)Y_k \\ &\leq \left(\hat{X}_k - C\right)^{-1} \leq \left(\hat{X}_L - C\right)^{-1} \end{aligned}$$

and $X_{k+1} = I_n - A^* Y_k^t A \geq I_n - A^*\left(\hat{X}_L - C\right)^{-t} A = X_L$.

Since $Y_k \leq \left(\hat{X}_{k-1} - C\right)^{-1} < \left(\hat{X}_k - C\right)^{-1}$, then

$$Y_k^{-1} < \hat{X}_k - C \text{,}$$

$$\begin{aligned} Y_{k+1} - Y_k &= Y_k - Y_k\left(\hat{X}_k - C\right)Y_k \\ &= Y_k\left(Y_k^{-1} - \left(\hat{X}_k - C\right)\right)Y_k < 0\ , \end{aligned}$$

By Lemma 2.3, we obtain

$$X_{k+1} - X_k = -A^*\left(Y_k^t - Y_{k-1}^t\right)A < 0\ ,$$

then $X_{k+1} < X_k$.

Thus by induction, we get that

$$X_0 > X_1 > X_2 > \cdots > X_k \geq X_L\ ,$$

$$Y_0 = Y_1 < Y_2 < \cdots < Y_{k-1} < Y_k \leq \left(\hat{X}_L - C\right)^{-1}\ ,$$

for arbitrary $k = 1,2,3,\cdots$ are all established.

Consequently, $\{X_n\}$ is monotonically decreasing and has lower bound, thus we have $X_n \to X_L$.

5 Numerical Examples

We consider the computational complexity of the algorithm 4.1 and 4.2 firstly. We might as well assume that all matrices are real matrix. For the matrix inversion operation of the algorithm 4.1, we generally use the Cholesky decomposition, i.e., $\left(\hat{X}_n - C\right)^t = LL^T$. Then, we solve the equation $LB = A$, and compute $X_{n+1} = Q - B^T B$. The Algorithm 4.2 avoids the matrix inverse operation, that is to say, every step of the calculation only needs matrix multiplication, and does not involve the inverse matrix calculation.

In this section, we give some numerical examples to illustrate the efficiency of the Algorithm 4.1 and Algorithm 4.2. All computations are performed on an Intel(R) Core(TM) i5-3210M CPU @ 2.50GHz computer. All the tests are performed by MATLAB, version 7.0.

Example 5.1

Considering the matrix equation

$$X + A^*\left(I_m \otimes X - C\right)^{-t} A = Q \tag{9}$$

where $m = 2$, $t = 0.8$,

$$A = \begin{pmatrix} 1 & 2 \\ 0.1 & 32 \\ 5 & 6 \\ 5 & 9 \end{pmatrix}, Q = \begin{pmatrix} 13 & 9 \\ 9 & 244 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By using Algorithm 4.1 and iterating forty-two steps, we obtain the maximal solution of Eq.(9) as follows:

$$X \approx X_{42} = \begin{pmatrix} 7.5870 & 2.1809 \\ 2.1809 & 221.7029 \end{pmatrix}.$$

with the residual

$$\left\| R\left(X_{42}\right) \right\|_{\mathrm{F}} = \left\| X_{42} - Q + A^{*}\left(I_{\mathrm{m}} \otimes X_{42} - C\right)^{-t} A \right\|_{F}$$
$$= 7.0278 \times 10^{-10}.$$

By using Algorithm 4.2 and iterating forty-three steps, we obtain the maximal solution of Eq.(9) as follows:

$$X \approx X_{43} = \begin{pmatrix} 7.5870 & 2.1809 \\ 2.1809 & 221.7029 \end{pmatrix}.$$

with the residual

$$\left\| R\left(X_{43}\right) \right\|_{F} = \left\| X_{43} - Q + A^{*}\left(I_{m} \otimes X_{43} - C\right)^{-t} A \right\|_{F}$$
$$= 6.382 \times 10^{-10}.$$

As can be seen from the Example 5.1, the algorithm 4.1 and algorithm 4.2 for solving the Eq.(9) is feasible and effective.

Example 5.2

Considering the matrix equation

$$X + A^{*}\left(I_{m} \otimes X - C\right)^{-t} A = Q, \tag{10}$$

where $m = 2$, $t = 1.5$,

$$A = \begin{pmatrix} 1 & 2 \\ 0.1 & 32 \\ 5 & 6 \\ 5 & 9 \end{pmatrix}, Q = \begin{pmatrix} 13 & 9 \\ 9 & 244 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By using Algorithm 4.1 and iterating ten steps, we obtain the maximal solution of Eq. (10) as follows:

$$X \approx X_{10} = \begin{pmatrix} 12.4161 & 8.3158 \\ 8.3158 & 242.9230 \end{pmatrix}.$$

with the residual

$$\left\| R\left(X_{10}\right) \right\|_{F} = \left\| X_{10} - Q + A^{*}\left(I_{m} \otimes X_{10} - C\right)^{-t} A \right\|_{F}$$
$$= 6.4855 \times 10^{-11}.$$

By using Algorithm 4.2 and iterating fifteen steps, we obtain the maximal solution of Eq.(10) as follows:

$$X \approx X_{15} = \begin{pmatrix} 12.4161 & 8.3158 \\ 8.3158 & 242.9230 \end{pmatrix}.$$

with the residual

$$\left\| R\left(X_{15}\right) \right\|_{F} = \left\| X_{15} - Q + A^{*}\left(I_{m} \otimes X_{15} - C\right)^{-t} A \right\|_{F}$$
$$= 1.3003 \times 10^{-11}.$$

As can be seen from the Example 5.2, the algorithm 4.1 and algorithm 4.2 for solving the Eq.(10) is feasible and effective. Although the algorithm 4.2 avoids the matrix inversion operation, the number of iterations is much more than that of algorithm 4.1.

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