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Quasi-equilibrium Problems and Fixed Point Theorems of the Product Mapping of Lower and Upper Semicontinuous Mappings

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Abstract In this paper, we introduce generalized quasi-equilibrium problems. These contain several problems in the optimization theory as special cases. We give sufficient conditions on the existence of their solutions. In particular, we establish several results on the existence of fixed points for product mappings of lower and upper semicontinuous mappings. These results generalize some well-known fixed point theorems obtained by previous authors as F. E. Browder and Ky Fan, X. Wu, L. J. Lin, and Z. T. Yu etc.

Keywords: Generalized quasi-equilibrium problems, upper and lower semi-continuous mappings, fixed point theorems

1 Introduction

It is well-known that the theory of fixed points plays an important role in applied mathematics. Many results in this theory become a tool to show the existence for solutions and to construct algorithms for finding solutions of many mathematical problems as optimization, variational, complementarity, equilibrium problems. We can shortly describe the main development of fixed point results of continuous mappings as follows. In 1911, L. Brouwer [1] used combinatorial method to show that a continuous mapping f from a simplex $K \subset R^n$ into itself has a fixed point, i. e, there exists a point $\bar{x} \in K$ such that $f(\bar{x}) = \bar{x}$. J. Schauder, 1930, [2] extended this result to the case that K is a nonempty convex compact subset in R^n. In 1941, S. Kakutani [3] generalized to the case when f is a upper semi-continuous mapping with nonempty convex and closed values from K to itself in R^n. In 1967, Ky Fan [4] proved a fixed point theorem of upper semi-continuous mappings with nonempty convex and closed values from a nonempty convex and compact subset K into itself in Hausdorff topological locally convex spaces. In 1968 F. E. Browder and Ky Fan [5]obtained a fixed point theorem of multivalued mapping which has open lower sections. Recently, many authors studied fixed point theorems of lower semi-continuous multivalued mappings with nonempty convex closed values, by using a continuous selection theorem, see for example, N. C. Yannelis and N. D. Prabhakar [6], Ben-El-Mechaiekh [7], X P. Ding, W. K. Kim and K. K. Tan [8], C. D. Horvath [9], X. Wu [10], S. Park [11],Z. T. Yu and L. J. Lin [12] and many others. In particular, Wu [10] obtained the following result.

Theorem ([16]) . *Let X be a nonempty subset of Hausdorff locally convex topological vector space, let D be a nonempty compact metrizable subset of X and let $T : X \to 2^D$ be a multivalued mapping with the following properties:*
$(i)\, T(x)$ is a nonempty convex closed set for each $x \in X$;
$(ii)\, T$ is lower semi-continuous.
Then there exists a point $x \in D$ such that $x \in T(x)$.

In this paper, we first establish a theorem on the existence of quasi-equilibrium points of multivalued mappings defined on subsets of Hausdorff locally convex topological vector spaces as follows.

Let X, Y be Hausdorff locally convex topological vector spaces over reals, $D \subset X$ be a nonempty subset. Given multi-valued mappings $P : D \to 2^D$ and $F : D \to 2^Y$, we are interested in the problem, denoted by (QEP), of finding $\bar{x} \in D$ such that

$$\bar{x} \in P(\bar{x});$$
$$0 \in F(\bar{x}).$$

This problem is called a quasi-equilibrium problem in which the multivalued mapping P is a constraint mapping and F is a utility multivalued mapping that are often determined by equalities and inequalities, or by inclusions and intersections of other multi-valued mappings, or by general relations in product spaces. The existence of solutions to this problem is studied in [[13], [14]] for the case the multivalued mapping P is continuous, and the multivalued mapping F is upper semicontinuous. All these mapping P and F need to have nonempty convex and closed values.

As far as we know equilibrium problems as generalizations of variational inequalities and optimization problems, including also optimization-related problems such as fixed point, complementarity problems, Nash equilibrium, minimax problems, etc. For the last decade there has been a number of generalizations of these problems to different directions such as quasi-equilibrium problems with constraint sets depending on parameters, quasi-variational and quasi-equilibrium inclusion problems with multi-valued data (see, for examples, in [8],[13], [14], [15]). Problem (QEP) described above is quite general. It encompasses a large class of problems of applied mathematics including quasi-optimization problems, quasi-variational inclusion, quasi-equilibrium problems, quasi-variational relation problems etc. Typical instances of (QEP) are shown in [13], [14] and [15] involving upper semi-continuous utility multivalued mappings with nonempty convex closed values. In this paper, we consider the above (QEP) in the product spaces as follows.

Let X, Y and Z be Hausdorff locally convex topological vector spaces over reals, $D \subset X, K \subset Z$ be nonempty subsets. Given multi-valued mappings $P : D \times K \to 2^D, Q : D \times K \to 2^K$ and $G : D \times K \to 2^X, H : D \times K \to 2^Z$ we are interested in the problem, denoted by (QEP), of finding $(\bar{x}, \bar{y}) \in D \times K$ such that

$$(\bar{x}, \bar{y}) \in P(\bar{x}, \bar{y}) \times Q(\bar{x}, \bar{y});$$
$$0 \in G(\bar{x}, \bar{y}) \times H(\bar{x}, \bar{y}).$$

Theorem 1 in Section 3 below shows sufficient conditions for the existence of solutions to this problem with different continuous assumptions on the multivalued mappings P, Q, G and H.

2 Preliminaries and Definitions

Throughout this paper, as mentioned in the introduction, X, Y and Z are real Hausdorff topological vector spaces, R is the space of real numbers, $R^* = R \cup \{\pm\infty\}$. Given a subset $D \subset X$, we consider a multivalued mapping $F : D \to 2^Y$. Let $F^{-1} : Y \to 2^X$ be defined by the condition that $x \in F^{-1}(y)$ if and only if $y \in F(x)$. We recall that

(a) The domain and the graph of F are denoted by

$$domF = \{x \in D | F(x) \neq \emptyset\},$$

$$Gr(F) = \{(x, y) \in D \times Y | y \in F(x)\},$$

respectively;

(b) F is said to be a closed mapping if the graph $Gr(F)$ of F is a closed subset in the product space $X \times Y$;

(c) F is said to be a compact mapping if the closure $\overline{F(D)}$ of its range $F(D)$ is a compact set in Y;

(d) $F : D \to 2^Y$ is said to be upper semi-continuous (in short, u.s.c) at $\bar{x} \in D$ if for each open set V containing $F(\bar{x})$, there exists an open set U of $\bar{x}$ such that for each $x \in U, F(x) \subset V$. F is said to be u.s.c on D if it is u.s.c at all $x \in D$;

(e) F is said to be lower semi-continuous (in short, l.s.c) at $\bar{x} \in D$ if for any open set V with $F(\bar{x}) \cap V \neq \emptyset$, there exists an open set U containing $\bar{x}$ such that for each $x \in U, F(x) \cap V \neq \emptyset$. F is said to be l.s.c on D if it is l.s.c at all $x \in D$;

(f) F is said to be continuous on D if it is at the same time u.s.c and l.s.c on D;

(g) F is said to have open lower sections if the inverse mapping F^{-1} is open valued, i.e„ for all $y \in Y, F^{-1}(y)$ is open in X.

Proposition 1. *$F : D \to 2^Y$ is l.s.c at $x \in D, F(x) \neq \emptyset$, if and only if for any net $\{x_\alpha\}$ in $D, x_\alpha \to x, y \in F(x)$, there is a net $\{y_\alpha\}$ with $y_\alpha \in F(x_\alpha), y_\alpha \to y$.*

Proof. Let F is l.s.c at x and $x_\alpha \to x$ and $y \in F(x)$. For arbitrary neighborhood V of the origin in Y, there exists α_0 such that $F(x_\alpha)\cap(y+V) \neq \emptyset$, for all $\alpha \geq \alpha_0$. Therefore, we can choose $y_\alpha \in F(x_\alpha)\cap(y+V)$. Thus, we have $y_\alpha - y \in V$, for all $\alpha \geq \alpha_0$. This shows $y_\alpha \to y$. Conversely, let $x \in D$ and N be an open subset such that $F(x) \cap N \neq \emptyset$. We assume that F is not l.s.c. at x. Then, there is an open subset N in Y with $F(x) \cap N \neq \emptyset$ such that for any neighborhood U_α of x there exists $x_\alpha \in U_\alpha$ such that $F(x_\alpha) \cap N = \emptyset$. This follows $F(x_\alpha) \subseteq Y \setminus N$, a closed set. Without loss of generality, we may maxpose that $x_\alpha \to x$. If $y_\alpha \in F(x_\alpha), y_\alpha \to y$, we deduce $y \in Y \setminus N$ and so $y \notin N$. Thus, $x_\alpha \to x$ and for any $y \in F(x)$, we can not find any $y_\alpha \in F(x_\alpha)$ with $y_\alpha \to y$. And, we have the proof of the converse part.

By X^* we denote the dual space of X i.e.,

$$X^* = \{f : X \to R | f \text{ is a linear and continuous function }\}.$$

The pairing $< .,. >$ between elements of $p \in X^*$ and $x \in X$ is defined by $< p, x >= p(x)$. We have

Proposition 2. *Asume that $F : D \to 2^Y$ is a l.s.c (u.s.c) multivalued mapping with nonempty values on D and $p \in X^* >$. Then, the function $c_p : D \to R^*$, defined by $c_p(x) = \inf_{v \in F(x)} < p, v >$ (respectively, $c_p(x) = \sup_{v \in F(x)} < p, v >$) is upper semi-continuous on D.*

Proof. Let $x \in D, \{x_\alpha\}$ be a net in D and $x_\alpha \to x$. Given $\epsilon > 0$, we take a neighborhood V of the origin in X such that $| < p, v > | < \epsilon$, for all $v \in V$. For $y \in F(x)$, we have $F(x) \cap (y + V) \neq \emptyset$. The lower semi-continuity of F implies that there exists α_0 such that $F(x_\alpha) \cap (y + V) \neq \emptyset$ with $\alpha > \alpha_0$. Therefore, we can take $y_\alpha \in F(x_\alpha) \cap (y + V), y_\alpha = y + v$, with $v \in V$, or $y = y_\alpha - v \in F(x_\alpha) + V$. This follows

$$< p, y >=< p, y_\alpha - v >\geq \inf_{w \in F(x_\alpha)+V} < p, w >\geq$$

$$\inf_{w \in F(x_\alpha)} < p, w > + \inf_{w \in V} < p, w >\geq \inf_{w \in F(x_\alpha)} < p, w > -\epsilon = c_p(x_\alpha) - \epsilon.$$

Taking $\lim_\alpha$ both the sides, we conclude

$$< p, y >\geq \lim_\alpha c_p(x_\alpha) - \epsilon.$$

This gives

$$c_p(x) \geq \lim_\alpha c_p(x_\alpha).$$

Thus, the function $c_p(.)$ is upper semi-continuous and the proof for the rest assertion is analogous.

Proposition 3. *Let $F : D \to 2^Y$ be a multivalued mapping with nonempty values on D. Then, if F has open lower sections, then F is l.s.c on D.*

Proof. Let $x \in D$ and N be an open subset in Y with $F(x) \cap N \neq \emptyset$. We take $y \in F(x) \cap N$. Then, $x \in F^{-1}(y)$. Since this set is open, then there exists a neighborhood U of x such that $x \in U \subset F^{-1}(y)$. This follows $x' \in F^{-1}(y)$ for all $x' \in U$, and hence $y \in F(x') \cap N$. Therefore, $F(x') \cap N \neq \emptyset$, for all $x' \in U$. Thus, F is l.s.c. on D. The proof of the proposition is completed.

It is easy to give examples proving that a continuous mapping may not have open lower sections.

Proposition 4. *Let $F_i : D \to 2^Y, i = 1, 2,$ be a l.s.c multivalued mapping with nonempty values on D. Then, the multivalued mapping $F : D \to 2^Y$ defined by*

$$F(x) = (F_1 + F_2)(x) = F_1(x) + F_2(x), x \in D,$$

is also l.s.c on D.

Proof. The proof is trivial by using Proposition 2.

Proposition 5. *Let $F_i : D \to 2^Y, i = 1, 2,$ be multivalued mappings with nonempty values on D. Assume that F_1 is l.s.c and F_2 has open lower sections. Then, the multivalued mapping $F : D \to 2^Y$ defined by*

$$F(x) = F_1(x) \cap F_2(x), x \in D,$$

is l.s.c on D.

Proof. Let $x \in D$ and N be an open subset in Y such that $F_1(x) \cap F_2(x) \cap N \neq \emptyset$. We take y in this set. Since N is open, we can choose an open neighborhood V of the origin in Y such that $y + V \subset N$. For $y \in F_2(x)$ and F_2 has open lower sections, one can find a neighborhood U_1 of x such that $x \in U \subset F_2^{-1}(y)$ Hence, $y \in F_2(x')$, for any $x' \in U_1$. Further, since $y \in F_1(x) \cap (y + V)$ and F_1 is l.s.c at x, then there is a neighborhood U_2 such that $F_1(x') \cap (y + V) \neq \emptyset$, for all $x' \in U_2$ and so $y \in F_1(x') \cap N$, for all $x' \in U_2$. Setting $U = U_1 \cap U_2$, we conclude that $y \in F_1(x') \cap F_2(x') \cap N \neq \emptyset$, for all $x' \in U$. This shows that $F = F_1 \cap F_2$ is l.s.c at x. Thus, the proof of the proposition is completed.

Proposition 6. *Let $F : D \to 2^Y$ be a multivalued mapping with nonempty values on D. If F has open lower sections, then the multivalued mapping $coF : D \to 2^Y$, defined by $(coF)(x) = coF(x)$, with $co(A)$ denoting the convex hull of A, also has open lower sections.*

Proof. Let $y \in Y$ and $x \in D$ with $y \in (coF)(x)$. We can write $y = \sum_{i=1}^{n} \alpha_i y_i$ with $\alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1, y_i \in F(x)$. This follows $x \in F^{-1}(y_i)$ Since F has open lower sections, there exists U_i such that $x \in U_i \subseteq F(y_i)$. for $i = 1, ..., n$ Taking $U = \cap_{i=1}^{n} U_i$, we can see $y_i \in F(x')$ for all $x' \in U$ and $i = 1, ..., n$. Therefore, $y = \sum_{i=1}^{n} \alpha_i y_i \in coF(x')$ for all $x' \in U$ and so $x \in U \subseteq (coF)^{-1}(y)$. This shows that coF has open lower sections. The proof of the proposition is completed.

Proposition 7. *Let $F : D \to 2^Y$ be a l.s.c multivalued mapping with nonempty values on D. Then so is the multivalued mapping $coF : D \to 2^Y$, defined by $(coF)(x) = coF(x)$.*

Proof. Indeed, let $x, x_\alpha \in D, x_\alpha \to x$ and $y \in (coF)(x), y = \sum_{i=1}^{m} \alpha_i y^i$ with $\alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1$ and $y^i \in F(x)$. Since F is l.s.c, there exist $y_\alpha^i \in F(x_\alpha), y_\alpha^i \to y^i$. Taking $y_\alpha = \sum_{i=1}^{m} \alpha_i y_\alpha^i$, we can see $y_\alpha \in (coF)(x_\alpha)$ and $y_\alpha \to y$.

The proof of the proposition is completed.

The following theorem is very important in the proof of the main result in this paper.

Theorem 8. ([16]). *Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be a an open cover of locally compact Hausdorff space X, $D \subset X$ be a compact set. Then, there exist continuous functions $\psi_i : D \to R, (i = 0, 1, ..., s)$ such that*
(i) $0 \leq \psi_i(x) \leq 1$;
(ii) $\sum_{i=1}^{s} \psi_i(x) = 1,$ *for all* $x \in D$;
(iii) For any $i \in \{0, 1, ..., s\}$, there exists $\alpha \in \Lambda$ such that $supp\psi_i \subset V_\alpha$, where $supp\psi = \{x \in D | \psi(x) \neq 0\}$.

The system of functions $\{\psi_i\}, i = 0, 1, ..., s,$ is said to be a partition of unity corresponding to the open cover $\{V_\alpha\}$.

3 Main Results

In this section we shall apply Theorem 8 in Section 2 above on partition of unity and our result in [13] to obtain sufficient conditions for solutions of (QEP). Before proving the main results in this section, we recall the following notions. Let D be a subset in X and $x \in D$. The set

$$T_D(x) = \overline{\{\alpha(y-x), y \in D, \alpha \geq 0\}} = \overline{\{cone(D-x)\}},$$

is called the tangent cone to the set D at x, where $coneM = \{\alpha z, z \in M, \alpha \geq 0\}$.

We now prove the following theorem on the existence for solutions of the above quasi-equilibrium problems concerning separately l.s.c. and u.s.c multivalued mappings.

Theorem 1. *We assume that the following conditions hold:*

(i) D, K *are nonempty convex compact sets;*
(ii) $P : D \times K \to 2^D$ *is a continuous multivalued mapping with nonempty closed convex values;*
(iii) $Q : D \times K \to 2^K$ *is a u.s.c multivalued mapping with nonempty closed convex values;*
(iv) $G : D \times K \to 2^X$ *is a l.s.c multivalued mapping;*
(v) $H : D \times K \to 2^Z$ *is a u.s.c multivalued mapping;*
(vi) For any $(x, y) \in P(x, y) \times Q(x, y), G(x, y)$ *is nonempty,* $H(x, y)$ *is nonempty convex closed and* $G(x, y) \subset T_{P(x,y)}(x), H(x, y) \cap T_{Q(x,y)}(y) \neq \emptyset$.

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ *such that*

$$(\bar{x}, \bar{y}) \in P(\bar{x}, \bar{y}) \times Q(\bar{x}, \bar{y});$$
$$0 \in G(\bar{x}, \bar{y}) \times H(\bar{x}, \bar{y}).$$

Proof. We set

$$B = \{(x, y) \in D \times K | x \in P(x, y), \ y \in Q(x, y)\}.$$

Since the multivalued mapping $S : D \times K \to 2^{D \times K}$, defined by

$$S(x, y) = P(x, y) \times Q(x, y), \ (x, y) \in D \times K,$$

is upper semi-continuous with nonempty convex and compact values, by using Ky Fan fixed point Theorem, we conclude that S has a fixed point in $D \times K$. Therefore, B is a nonempty set. The upper semi-continuity and the closedness of values of S imply that B is a closed and then compact set.

Assume that for any $(x, y) \in B$, $0 \notin G(x, y) \times H(x, y)$. Take a fixed $v \in G(x, y)$. Remarking that $H(x, y)$ is a nonempty closed convex, so is the set $\{v\} \times H(x, y)$. Since $0 \notin \{v\} \times H(x, y)$, by Hahn -Banach Theorem, there exists $p \in (X \times Z)^*$ such that

$$\beta = \sup_{(v,w) \in \{v\} \times H(x,y)} p(v, w) < 0.$$

We have

$$p(v, 0) + p(0, w) \leq p(v, 0) + \sup_{w \in H(x,y)} p(0, w) < 0.$$

This gives

$$\inf_{v \in co(G(x,y))} p(v, 0) + \sup_{w \in H(x,y)} p(0, w) < 0.$$

Further, we define functions ${c_p}^1(.,.) : D \times K \to R^*, {c_p}^2(.,.) : D \times K \to R^*$ by

$${c_p}^1(x, y) = \inf_{v \in co(G(x,y))} p(v, 0);$$

$$c_p{}^2(x,y) = \sup_{w\in H(x,y)} p(0,w).$$

By Proposition 2 in Section 2, these functions are u.s.c on $D\times K$, therefore, the set

$$U_p(x,y) = \{(x',y')\in D\times K | c_p{}^1(x',y') + c_p{}^2(x',y') < 0\}$$

is a nonempty open neighborhood of (x,y).
Thus, for any $(x,y)\in B$ there is $p\in (X\times Z)^*$ such that

$$U_p(x,y) = \{(x',y')\in D\times K | c_p^1(x',y') + c_p^2(x',y') < 0\}$$

is nonempty and open and hence $\{U_p(x,y)\}_{p\in(X\times Z)^*}, (x,y)\in B$, is an open cover of B. Since B is compact, there exist finite $p_1,...,p_s\in (X\times Z)^*$ such that

Further, since B is closed in $D\times K, U_{p_0} = D\times K\setminus B$ is open in $D\times K$ and hence $\{U_{p_0}, U_{p_1},...,U_{p_s}\}$ is an open cover of the compact set $D\times K$. By Theorem 8 in Section 2, there exist continuous functions $\psi_i : D\times K\to R, (i=0,1,...,s)$ such that
$(i)\, 0\le \psi_i(x,y)\le 1$;
$(ii)\, \sum\limits_{i=1}^{s}\psi_i(x,y) = 1,$ for all $(x,y)\in D\times K$;
(iii) For any $i\in\{0,1,...,s\}$, there exists $j(i)\in\{0,...,s\}$ such that $supp\psi_i\subset U_{p_{j(i)}}$. It is clear that $supp\psi_0\subset U_{p_0}\subset D\times K\setminus B$.
Further, we set $\tilde{D} = \tilde{K} = D\times K$ and define the function $\phi : \tilde{K}\times\tilde{D}\times\tilde{D}\to R$ by

$$\phi(((v,y),(x,w),(t,z)) = \sum_{i=0}^{s}\psi_i(x,y).(p_{j(i)}(t-x,0) + p_{j(i)}(0,z-y)), (v,y),(x,w),(t,z)\in D\times K.$$

Then, ϕ is a continuous function on $\tilde{K}\times\tilde{D}\times\tilde{D}$. Moreover, for any fixed $(v,y)\in D\times K, \phi((v,y),(x,w),.) : \tilde{D}\to R$ is a linear function and $\phi((v,y),(t,z),(t,z)) = 0$ for all $(v,y)(t,z),\in D\times K$. Therefore, $\tilde{D},\tilde{K},\tilde{P}((x,w),(v,y)) = P(x,y)\times K, \tilde{Q}((x,w),(v,y)) = D\times Q(x,y)$ and ϕ satisfy all conditions of Corollary 3.4 in [13]. It implies that there is $(\bar{x},\bar{w}),(\bar{v},\bar{y})\in\tilde{D}\times\tilde{K}$ such that $(\bar{x},\bar{w})\in\tilde{P}(\bar{x},\bar{w}),(\bar{v},\bar{y})),(\bar{v},\bar{y})\in\tilde{Q}(\bar{x},\bar{w}),(\bar{v},\bar{y}))$ and $\phi((\bar{v},\bar{y}),(\bar{x},\bar{w}),(t,z))\ge 0$,for all $(t,z)\in\tilde{P}((\bar{x},\bar{w})(\bar{v},\bar{y}))$. This gives

$$\sum_{i=0}^{s}\psi_i(\bar{x},\bar{y}).(p_{j(i)}(t-\bar{x},0) + p_{j(i)}(0,z-\bar{y}))\ge 0 \text{ for all } (t,z)\in\tilde{P}(\bar{x},\bar{w}),(\bar{v},\bar{y})). \tag{3.1}$$

Setting $p^* = \sum\limits_{i=0}^{s}\psi_i(\bar{x},\bar{y}).p_{j(i)}$, we get from (3.1) $(p^*(t-\bar{x},0) + p^*(0,z-\bar{y}) >\ge 0$, for all $(t,z)\in P(\bar{x},\bar{y})\times K$, and hence $p^*(v,0)+p^*(0,w)\ge 0$, for all $(v,w)\in T_{P(\bar{x},\bar{y})}(\bar{x})\times T_K(\bar{y})$. By Assumption $(v), G(\bar{x},\bar{y})\subset T_{P(\bar{x},\bar{y})}(\bar{x}); H(\bar{x},\bar{y})\cap T_{Q(\bar{x},\bar{y})}(\bar{y})\ne\emptyset; T_{Q(\bar{x},\bar{y})}(\bar{y})\subset T_K(y)$, we conclude that for any $(v,w)\in co(G(\bar{x},\bar{y}))\times(H(\bar{x},\bar{y})\cap T_{Q(\bar{x},\bar{y})}(\bar{y}))$ it holds

$$p^*(v,w) = p^*(v,0) + p^*(0,w)\ge 0.$$

This follows

$$\inf_{(v,w)\in co(G(\bar{x},\bar{y})\times(H(\bar{x},\bar{y})\cap T_{Q(\bar{x},\bar{y})}(\bar{y}))} p^*(v,w)\ge 0. \tag{3.2}$$

Further, put $I(\bar{x},\bar{y}) = \{i\in\{0,1,...,s\}|\psi_i(\bar{x},\bar{y}) > 0\}$. Since $\psi_i(\bar{x},\bar{y})\ge 0$ and $\sum\limits_{i=1}^{s}\psi_i(x,y) = 1$, we deduce $I(\bar{x},\bar{y})\ne\emptyset$. So, for any $i\in I(\bar{x},\bar{y}), (\bar{x},\bar{y})\in supp\psi_i\subset U_{p_{j(i)}}$ and remarking $(\bar{x},\bar{y})\in B$,

$$c^1_{p_{j(i)}}(\bar{x},\bar{y}) = \inf_{v\in co(G(\bar{x},\bar{y}))} p_{j(i)}(v',0);$$

$$c^2_{p_{j(i)}}(\bar{x},\bar{y}) = \sup_{w'\in H(\bar{x},\bar{y})} p_{j(i)}(0,w')$$

and

$$c^1_{p_{j(i)}}(\bar{x},\bar{y}) + c^2_{p_{j(i)}}(\bar{x},\bar{y}) < 0\ . \tag{3.3}$$

For any $(v,w) \in co(G(\bar{x},\bar{y})) \times (H(\bar{x},\bar{y}) \cap T_{Q(\bar{x},\bar{y})}(\bar{y}))$, we have

$$p^*(v,0) + p^*(0,w) = \sum_{i=0}^{s} \psi_i(\bar{x},\bar{y}).\{p_{j(i)}(v,0) + p_{j(i)}(0,w)\}$$

$$\leq \sum_{i=0}^{s} \psi_i(\bar{x},\bar{y}) \max_{i=1,\ldots,s}\{p_{j(i)}(v,0) + p_{j(i)}(0,w)\} \leq \max_{i=1,\ldots,s}\{p_{j(i)}(v,0) + p_{j(i)}(0,w)\}.$$

Hence,

$$\inf_{(v,w)\in co(G(\bar{x},\bar{y})\times(H(\bar{x},\bar{y})\cap T_{Q(\bar{x},\bar{y})}(\bar{y}))} p^*(v,w) = \inf_{(v,w)\in co(G(\bar{x},\bar{y}))\times(H(\bar{x},\bar{y})\cap T_{Q(\bar{x},\bar{y})}(\bar{y}))}\{p^*(v,0) + p^*(0,w)\}$$

$$\leq \inf_{(v,w)\in co(G(\bar{x},\bar{y}))\times(H(\bar{x},\bar{y})\cap T_{Q(\bar{x},\bar{y})}(\bar{y}))} \max_{i=1,\ldots,s}\{p_{j(i)}(v,0) + p_{j(i)}(0,w)\}. \tag{3.4}$$

Setting $C = \bar{co}\{p_{j(1)},...,p_{j(s)}\}, E = co(G(\bar{x},\bar{y})) \times (H(\bar{x},\bar{y}) \cap T_{Q(\bar{x},\bar{y})}(\bar{y})), f(p,u) = p(v,0) - p(0,w),\ u = (v,w)$ and using the weak* topology on $(X \times Z)^*$, we can easily verify that all conditions of Sion's minimax Theorem in [17] are satisfied. Therefore, we obtain

$$\inf_{(v,w)\in co(G(\bar{x},\bar{y}))\times(H(\bar{x},\bar{y})\cap T_{Q(\bar{x},\bar{y})}(\bar{y}))} \max_{i=1,\ldots,s}\{p_{j(i)}(v,0) + p_{j(i)}(0,w)\}$$

$$= \max_{i=1,\ldots,s} \inf_{(v,w)\in co(G(\bar{x},\bar{y}))\times H(\bar{x},\bar{y})\cap T_{Q(\bar{x},\bar{y})}(\bar{y})} p_{j(i)}(v,w)$$

$$\leq \max_{i=1,\ldots,s}\{ \inf_{v\in co(G(\bar{x},\bar{y}))} p_{j(i)}(v,0) + \sup_{w\in(H(\bar{x},\bar{y})\cap T_{Q(\bar{x},\bar{y})}(\bar{y}))} p_{j(i)}(0,w)\}$$

$$\leq \max_{i=1,\ldots,s}\{c^1_{p_{j(i)}}(\bar{x},\bar{y}) + c^2_{p_{j(i)}}(\bar{x},\bar{y})\} < 0\ . \tag{3.5}$$

A combination of (3.4) and (3.5) implies

$$\inf_{(v,w)\in co(G(\bar{x},\bar{y}))\times(H(\bar{x},\bar{y})\cap T_{Q(\bar{x},\bar{y})}(\bar{y}))} p^*(v,w) < 0.$$

Thus, we have a contradiction to (3.2).
This completes the proof of the theorem.

In Particular, we obtain the fixed point result.

Corollary 2. *We assume that the following conditions hold:*

(i) D, K *are nonempty convex compact sets;*
(ii) $P : D \times K \to 2^D$ *is a continuous multivalued mapping with nonempty closed convex values;*
(iii) $Q : D \times K \to 2^K$ *is a u.s.c multivalued mapping with nonempty closed convex values;*
(iv) $G : D \times K \to 2^X$ *is a l.s.c multivalued mapping ;*
(v) $H : D \times K \to 2^Z$ *is a u.s.c multivalued mapping;*
(vi) For any $(x,y) \in P(x,y) \times Q(x,y), G(x,y)$ *is nonempty,* $H(x,y)$ *is nonempty convex closed and* $G(x,y) - x \subset T_{P(x,y)}(x), (H(x,y) - y) \cap T_{Q(x,y)}(y) \neq \emptyset$.

Then there exists $(\bar{x},\bar{y}) \in D \times K$ *such that*

$$(\bar{x},\bar{y}) \in (P(\bar{x},\bar{y}) \times Q(\bar{x},\bar{y})) \cap (G(\bar{x},\bar{y}) \times H(\bar{x},\bar{y})).$$

Proof. We define the multivalued mappings $\tilde{G} : D \times K \to 2^X; \tilde{H} : D \times K \to 2^Z$ by

$$\tilde{G}(x,y) = G(x,y) - x\ ;$$
$$\tilde{H}(x,y) = H(x,y) - y,\ (x,y) \in D \times K.$$

Remarking that by Propositon 4 in Section 2 $\tilde{G}$ is l.s.c.$\tilde{H}$ is u.s.c and $\tilde{G}(x,y) \neq \emptyset, \tilde{H}(x,y)$ is nonempty closed and convex $\tilde{G}(x,y) \subset T_{P(x,y)}(x); \tilde{H}(x,y) \cap T_{Q(x,y)}(y) \neq \emptyset$ for any $(x,y) \in P(x,y) \times Q(x,y)$. Further, the proof of this corollary follows immediately from Theorem 1.

Corollary 3. *We assume that the following conditions hold:*

(i) D, K *are nonempty convex compact sets;*
(ii) $P : D \times K \to 2^D$ *is a continuous multivalued mapping with nonempty closed convex values;*
(iii) $Q : D \times K \to 2^K$ *is a u.s.c multivalued mapping with nonempty closed convex values;*
(iv) $G : D \times K \to 2^X$ *is a l.s.c multivalued mapping;*
(v) For any $(x, y) \in P(x, y) \times Q(x, y), x \notin G(x, y),$ *and* $G(x, y) - x \subset T_{P(x,y)}(x)$.

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ *such that*
1) $\bar{x} \in P(\bar{x}, \bar{y})$;
2) $\bar{y} \in Q(\bar{x}, \bar{y})$;
3) $G(\bar{x}, \bar{y}) = \emptyset$.

Proof. We assume that $G(x, y) \neq \emptyset$, for all $(x, y) \in P(x, y) \times Q(x, y)$. We define the multivalued mapping $\tilde{G} : D \times K \to 2^X, \tilde{H} : D \times K \to 2^Z$ by

$$\tilde{G}(x, y) = G(x, y) - x, \ ;$$

$$\tilde{H}(x, y) = y, (y, x) \in D \times K.$$

Then $\tilde{G}(x, y) \neq \emptyset$ for all $(x, y) \in P(x, y) \times Q(x, y)$ and $\tilde{G}(x, y) \subset T_{P(x,y)}(x), \tilde{H}(x, y) - y = 0 \in T_{Q(x,y)}(y)$. Using Proposition 4 in Section 2, we conclude that $\tilde{G}$ is a l.s.c multivalued mapping and $\tilde{G}(x, y) \neq \emptyset$ for all $(x, y) \in P(x, y) \times Q(x, y)$. Further, the proof of this corollary follows immediately from Corollary 2 to obtain $(\bar{x}, \bar{y}) \in D \times K$ such that
1) $\bar{x} \in P(\bar{x}, \bar{y})$;
2) $\bar{y} \in Q(\bar{x}, \bar{y})$;
3) $\bar{x} \in G(\bar{x}, \bar{y})$. Thus, we have a contradiction and the proof of the corollary is complete.

Corollary 4. *We assume that the following conditions hold:*

(i) D, K *are nonempty convex compact sets;*
(ii) $P : D \times K \to 2^D$ *is a continuous multivalued mapping with nonempty closed convex values;*
(iii) $Q : D \times K \to 2^K$ *is a u.s.c multivalued mapping with nonempty closed convex values;*
(iv) $H : D \times K \to 2^Z$ *is a u.s.c multivalued mapping with nonempty convex compact values;*
(v) For any $(x, y) \in P(x, y) \times Q(x, y), y \notin H(x, y),$ *and* $(H(x, y) - y) \cap T_{Q(x,y)}(y) \neq \emptyset$.

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ *such that*
1) $\bar{x} \in P(\bar{x}, \bar{y})$;
2) $\bar{y} \in Q(\bar{x}, \bar{y})$;
3) $H(\bar{x}, \bar{y}) = \emptyset$.

Proof. We assume that $H(x, y) \neq \emptyset$, for all $(x, y) \in P(x, y) \times Q(x, y)$. We define the multivalued mapping $\tilde{G} : D \times K \to 2^X, \tilde{H} : D \times K \to 2^Y$ by

$$\tilde{G}(x, y) = x,$$

$$\tilde{H}(x, y) = H(x, y) - y, \ (y, x) \in D \times K.$$

Then $\tilde{H}(x, y) \neq \emptyset$ for all $(x, y) \in P(x, y) \times Q(x, y)$ and $\tilde{H}(x, y) \subset T_{Q(x,y)}(y) \neq \emptyset$. Using Proposition 4 in Section 2, we conclude that $\tilde{H}$ is a u.s.c multivalued mapping with nonempty convex compact values and $\tilde{H}(x, y) \neq \emptyset$ for all $(x, y) \in P(x, y) \times Q(x, y)$. Further, the proof of this corollary follows immediately from Corollary 2 to obtain $(\bar{x}, \bar{y}) \in D \times K$ such that
1) $\bar{x} \in P(\bar{x}, \bar{y})$;
2) $\bar{y} \in Q(\bar{x}, \bar{y})$;
3) $\bar{x} \in H(\bar{x}, \bar{y})$. Thus, we have a contradiction and the proof of the corollary is complete.

Corollary 5. *We assume that the following conditions hold:*

(i) D, K *are nonempty convex compact sets;*
(ii) $Q : D \times K \to 2^K$ *is a u.s.c multivalued mapping with nonempty closed convex values;*
(iii) $G : D \times K \to 2^X$ *is a l.s.c multivalued mapping with* $G(x, y) \neq \emptyset$ *and* $G(x, y) - x \subseteq T_D(x)$*, for any* $x \in D, y \in Q(x, y)$*.*

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ *such that*
1) $\bar{y} \in Q(\bar{x}, \bar{y})$;
2) $\bar{x} \in G(\bar{x}, \bar{y})$.

Proof. The proof of this corollary follows immediately from Theorem 1 with taking $\tilde{G}(x, y) = G(x, y) - x, \tilde{H}(x, y) = y, P(x, y) = D$, for all $(x, y) \in D \times K$.

We have a fixed point result of separately l.s.c and u.s.c multivalued mappings with nonempty convex closed values. This is a generalization of Ky Fan's Theorem.

Corollary 6. *We assume that the following conditions hold:*
(i) *D is nonempty convex compact subsets of X ;*
(ii)*$P : D \to 2^D$ is a continuous multivalued mapping with nonempty values;*
(ii) *$G : D \to 2^X$ is a l.s.c multivalued mapping with nonempty values such that $G(x) \subset T_{P(x)}(x)$ for any $x \in P(x)$.*
Then there exists $\bar{x} \in D$ such that $\bar{x} \in G(\bar{x}) \cap P(\bar{x})$.

Proof. We put $\tilde{G}(x, y) = G(x); \tilde{P}(x, y) = P(x); \tilde{Q}(x, y) = \tilde{H}(x, y) = K$ for any $(x, y) \in D \times K$ and apply Corollary 2.

Corollary 7. *We assume that the following conditions hold:*
(i) *K is a nonempty convex compact subset of Z ;*
(ii)*$Q : K \to 2^K$ is a u.s.c multivalued mapping with nonempty closed convex values;*
(ii) *$H : K \to 2^K$ is a u.s.c multivalued mapping with nonempty closed convex values such that $H(y) \cap T_{Q(y)}(y) \neq \emptyset$ for any $y \in Q(y)$.*
Then there exists $\bar{y} \in K$ such that $\bar{y} \in H(\bar{y}) \cap Q(\bar{y})$.

Proof. We put $\tilde{G}(x, y) = K$; $\tilde{P}(x, y) = D$; $\tilde{Q}(x, y) = Q(y)$, $\tilde{H}(x, y) = H(y)$ for any $(x, y) \in D \times K$ and apply Corollary 2.

4 Some Applications

In this section we introduce some applications of the above results to consider the existence of solutions to mixed generalized quasi-equilibrium problems concerning l.s.c and u.s.c continuous multivalued mappings. We assume that $X, Z, Y, Y_i, i = 1, 2$, are real Hausdorff topological vector spaces, $D \subset X, K \subset Z$ are nonempty subsets . Given multivalued mappings $S : D \times K \to 2^D, T : D \times K \to 2^K$ and $F : K \times K \times K \times D \to 2^Y$, we are interested in the problem of finding $(\bar{x}, \bar{y}) \in D \times K$ such that
i) $\bar{x} \in S(\bar{x}, \bar{y})$;
ii) $\bar{y} \in T(\bar{x}, \bar{y})$;
iii) $0 \in F(\bar{y}, \bar{y}, v, \bar{x})$, for all $v \in T(\bar{x}, \bar{y})$.
This problem is called a generalized quasi-equilibrium problem of type I , denoted by $(GEP)_I$.

Given multivalued mappings $P, P_0 : D \times K \to 2^D, Q : D \times K \to 2^K, Q_0 : K \times D \times D \to 2^K$, and $F : K \times K \times D \times D \to 2^Y$, we are interested in the problem of finding $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\bar{x} \in P(\bar{x}, \bar{y});$$

$$\bar{y} \in Q(\bar{x}, \bar{y})$$

and

$$0 \in F(\bar{y}, v, \bar{x}, t), \text{for all } t \in P_0(\bar{x}, \bar{y}) \text{ and } v \in Q_0(\bar{y}, \bar{x}, t).$$

This problem is called a generalized quasi-equilibrium problem of type II , denoted by $(GEP)_{II}$.

Further, given multivalued mappings $S : D \times D \to 2^D, T : D \times K \to 2^K, P_0 : D \times K \to 2^D, Q_0 : K \times D \times D \to 2^K$ and $F_1 : K \times K \times K \times D \to 2^{Y_1}, F_2 : K \times K \times D \times D \to 2^{Y_2}$, we are interested in the problem of finding $(\bar{x}, \bar{y}) \in D \times K$ such that
i) $\bar{x} \in S(\bar{x}, \bar{y})$;
ii) $\bar{y} \in T(\bar{x}, \bar{y})$;
iii) $0 \in F_1(\bar{y}, \bar{y}, v, \bar{x}), \text{for all } v \in T(\bar{x}, \bar{y})$;
iv) $0 \in F_2(\bar{y}, v, \bar{x}, t), \text{for all } t \in P_0(\bar{x}, \bar{y}), v \in Q_0(\bar{y}, \bar{x}, t)$.
This problem is called a mixed generalized quasi-equilibrium problem , denoted by $(MGQEP)$, in which the multivalued mappings S, T, P_0, Q_0 are called constraint mappings and F_1, F_2 are called utility multivalued mappings.
We apply the obtained results in Section 3 to get the existence to solutions for $(MGQEP)$ as follows.

Theorem 1. *The following conditions are sufficient for $(MGQEP)$ to have a solution:*

i) D and K are nonempty convex compact subsets;
ii) $P : D \times K \to 2^D$ is continuous multivalued mapping with nonempty convex closed values, $Q : D \times K \to 2^K$ is u.s.c. multivalued mapping with nonempty convex values;
iii) $P_0 : D \times K \to 2^D$ is a multivalued mapping with nonempty values and has open lower sections ; $coP_0(x, y) \subseteq P(x, y)$ for any $(x, y) \in D \times K$;
iv) The set $A = \{(y, w, v, x) \in K \times K \times K \times D | 0 \in F_1(y, w, v, x)\}$ is closed ;
v) For any fixed $(y, x) \in K \times K \times D$, the set $B = \{w \in T(x, y) | 0 \in F_1(y, w, v, x) \text{for all } v \in T(x, y)\}$ is nonempty convex.

vi) For any fixed $t \in D$, the set

$$A_1 = \{(x, y) \in D \times K | \ 0 \notin F_2(y, v, x, t), \ \textit{for some} \ \ v \in Q_0(y, x, t)\},$$

is open in D;
vii) For any fixed $y, v \in K, 0 \in F_2(y, v, x, x)$ for any $x \in D$.

Proof. We define the multivalued mapping $Q : D \times K \to 2^K$ by

$$H(x, y) = \{v \in Q(x, y) | 0 \in F_1(y, v, w, x), \text{ for all } \ w \in Q(x, y)\}.$$

Conditions v) and vi) imply that $H(x, y) \neq \emptyset$ and nonempty closed convex for any $(x, y) \in D \times K$. Now, let (x_α, y_α) be a net converging to (x, y) and w_α be a net with $w_\alpha \in H(x_\alpha, y_\alpha), w_\alpha \to w$. We have to show $w \in H(x, y)$. Indeed, we can see $0 \in F_1(y_\alpha, w_\alpha, u_\alpha, x_\alpha)$, for all $u_\alpha \in Q(x_\alpha, y_\alpha)$. Let $v \in Q(x, y)$ be arbitrary. Since Q is l.s.c, there is $v_\alpha \in Q(x_\alpha, y_\alpha), v_\alpha \to v$. Therefore, we get $0 \in F_1(y_\alpha, w_\alpha, v_\alpha, x_\alpha)$. For $(y_\alpha, w_\alpha, v_\alpha, x_\alpha) \to (y, w, v, x)$ and the set A is closed, we deduce $(y, w, v, x) \in A$. Hence, $0 \in F_1(y, w, v, x)$, for all $v \in Q(x, y)$. This shows that the multivalued mapping H is closed, and then H is u.s.c with nonempty closed convex values on $D \times K$.

Further, we define the multivalued mapping $G : D \times K \to K$ by

$$G(x, y) = \{t \in P_0(x, y) | 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(y, x, t)\}, \ (x, y) \in D \times K.$$

We can write $G(x, y) = T(x, y) \cap P_0(x, y)$, where

$$T(x, y) = \{t \in D | 0 \notin F_2(y, v, x, t) \text{ for some } v \in Q_0(y, x, t)\}.$$

For any $t \in D$, the set $T^{-1}(t) = A_1 = \{(x, y) \in D \times K | 0 \notin F(y, v, x, t) \text{ for some } v \in Q_0(y, x, t)\}$ is open in $D \times K$. So, the multivalued mapping T has open lower sections and then it is separately l.s.c. on $D \times K$.

For P_0 has open lower sections, we apply Propositions 5 and 7 in Section 2 to conclude that $G = T \cap P_0$ is a l.s.c multivalued mapping with nonempty values. We set

$$B = \{(x, y) \in D \times K | x \in P(x, y); y \in H(x, y)\}.$$

It is clear that B is a nonempty closed set in $D \times K$. Observe that if for some $(\bar{x}, \bar{y}) \in B$, it gives $G(\bar{x}, \bar{y}) \cap P_0(\bar{x}, \bar{y}) = \emptyset$, then
i) $\bar{x} \in P(\bar{x}, \bar{y})$;
ii) $\bar{y} \in Q(\bar{x}, \bar{y})$;
iii) $0 \in F_1(\bar{y}, \bar{y}, w, \bar{x})$, for all $w \in Q(\bar{x}, \bar{y})$.
vi) $0 \in F_2(\bar{y}, w, \bar{x}, t)$, for all $t \in P_0(\bar{x}, \bar{y})$ and $w \in Q_0(\bar{x}, t, \bar{y})$,
and hence the proof of the theorem is completed. Thus, our aim is to show the existence of such a point $(\bar{x}, \bar{y})$. Indeed, by contrary, we assume that for any $(x, y) \in B$, it implies that $G(x, y) \cap P_0(x, y) \neq \emptyset$. We consider the multivalued mapping $S : D \times K \to 2^{D \times K}$ defined by

$$S(x, y) = \begin{cases} G(x, y) \cap P_0(x, y) \times \{y\}, & \text{if } (x, y) \in B, \\ P_0(x, y) \times \{y\}, & \textit{else}. \end{cases}$$

We show that S verifies the hypotheses of Corollary 2 in Section 3. Applying this corollary, we conclude that there is a point $(\bar{x}, \bar{y}) \in D \times K$ such that $\bar{x} \in P(\bar{x}, \bar{y}), \bar{y} \in H(\bar{x}, \bar{y})$ and $\bar{x} \in S(\bar{x}, \bar{y})$). If $(\bar{x}, \bar{y}) \in B$, then $\bar{x} \in G(\bar{x}, \bar{y}) \cap P_0(\bar{x}, \bar{y})$ and hence $0 \notin F_2(\bar{y}, v, \bar{x}, \bar{x})$, for some $v \in Q_0(\bar{x}, \bar{x}, \bar{y})$, we have a contradiction to Condition vii). If $(\bar{x}, \bar{y}) \notin B$, then $(\bar{x}, \bar{y}) \in P_0(\bar{x}, \bar{y}) \times H(\bar{x}, \bar{y}) \subset P(\bar{x}, \bar{y}) \times H(\bar{x}, \bar{y}) = B$ and we also have a contradiction. The proof of the theorem is complete.

Corollary 2. *We assume that the following conditions hold:*

i) D and K are nonempty convex compact subsets;
ii) $P : D \times K \to 2^D, Q : D \times K \to 2^K$ are continuous multivalued mappings with nonempty convex values;
(iii) $\phi : K \times K \times D \times D \to R$ is a real function such that:
a) For any fixed $t \in D$, the function $\phi(., ., ., t) : K \times K \to R$ are upper semi-continuous;
b) $\phi(y, v, x, x) \geq 0$, for all $y, v \in K, x \in D$.

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that $(\bar{x}, \bar{y}) \in P(\bar{x}, \bar{y}) \times Q(\bar{x}, \bar{y})$ and

$$\phi(\bar{y}, v, \bar{x}, t) \geq 0, \textit{ for all } (t, v) \in P(\bar{x}, \bar{y}) \times Q(\bar{x}, \bar{y}).$$

Proof. We take $P_0 = P, Q_0 : K \times D \times D \to 2^K$ defined by $Q_0(y, x, t) = Q(x, y)$ and $F : K \times K \times D \times D \to 2^R$ defined by $F_2(y, v, x, t) = \phi(y, v, x, t) - R_+$. We verify that for any fixed $t \in D$, the set

$$A_1 = \{(x, y) \in D \times K |\ 0 \notin F_2(y, v, x, t), \text{ for some } v \in Q_0(y, x, t)\}$$
$$= \{(x, y) \in D \times K |\ \phi(y, v, x, t) < 0, \text{ for some } v \in Q(x, y)\}$$

is open in $D \times K$. Indeed,

$$(D \times K) \setminus A_1 = \{(x, y) \in D |\ \phi(y, v, x, t) \geq 0, \text{ for all } v \in Q(x, y)\}.$$

If $\{(x_\alpha, y_\alpha)\}$ is a net in $(D \times K) \setminus A_1, (x_\alpha, y_\alpha) \to (x, y)$ we have to show $(x, y) \in (D \times K) \setminus A_1$. Take arbitrary $v \in Q(x, y)$. Since $Q(., .) : D \to K$ is l.s.c., there is $v_\alpha \in Q(x_\alpha, y_\alpha), v_\alpha \to v$. For $(x_\alpha, y_\alpha) \in (D \times K) \setminus A_1$, it follows $\phi(y_\alpha, v_\alpha, x_\alpha, t) \geq 0$. The upper semicontinuity of $\phi(., ., ., t)$ implies that $\phi(y, v, x, t) \geq 0$. Thus, this shows that $(x, y) \in (D \times K) \setminus A_1$ and so, $(D \times K) \setminus A_1$ is closed and then A_1 is open. To complete the proof of the corollary, it remains to apply Theorem 1 with $P_0 = P, Q_0 : K \times D \times D \to 2^K$ defined by $Q_0(y, x, t) = Q(x, y)$ and F_2.

Conclusion. We prove Theorem 1 in Section 3 on the existence of solutions to quasi-equilibrium problems involving product mappings of lower and upper semicontinuous mappings. As corollaries, we obtain some fixed point results of product mappings. In Section 4, we introduce some applications of the above results to consider the existence of solutions to mixed generalized quasi-equilibrium problems concerning l.s.c and u.s.c continuous multivalued mappings.

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