Two-Dimensional Measure Vector of Departure from Marginal Homogeneity in Square Contingency Tables

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Abstract For square contingency tables with ordered categories, Tahata et al. (2006) and Tahata et al. (2012) considered measures which represent the degree of departure from the marginal homogeneity (MH) model using the cumulative marginal probabilities. The former measure can judge whether or not the MH model holds, but the latter cannot. Also, the latter measure can distinguish the directionality for two types of maximum marginal inhomogeneities, but the former cannot. The present paper proposes a two-dimensional measure vector with above measures as elements that can simultaneously analyze the degree of departure from MH and the directionality for two types of maximum marginal inhomogeneities. Also, this paper shows the usefulness of the proposed measure vector on comparing visually degrees of departure from MH in several square contingency tables using confidence region.

Keywords: Comparison, confidence region, ordinal category, visualized measure.

1 Introduction

Consider an $r \times r$ square contingency table with ordered categories. Let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table $(i = 1, \ldots, r; j = 1, \ldots, r)$, and let X and Y denote the row and column variables, respectively.

The symmetry (S) model (Bowker, 1948) is defined by

$$\Pr(X = i, Y = j) = \Pr(X = j, Y = i) \quad (i \neq j),$$

namely

$$p_{ij} = p_{ji} \quad (i \neq j).$$

For representing the degree of departure from the S model, Tomizawa, Miyamoto and Hatanaka (2001) proposed a measure γ (although the detail is omitted), and Tahata, Miyazawa and Tomizawa (2010) proposed a measure ϕ (although the detail is omitted). Also, Ando, Tahata and Tomizawa (2017) proposed a two-dimensional measure vector $\boldsymbol{\Phi} = (\gamma, \phi)^t$ to represent the degree of departure from the S model. Note that the symbol " t " denotes the transpose.

The marginal homogeneity (MH) model (Stuart, 1955) is defined by

$$\Pr(X = i) = \Pr(Y = i) \quad (i = 1, \ldots, r),$$

namely

$$p_{i \cdot} = p_{\cdot i} \quad (i = 1, \ldots, r),$$

where

$$p_{i \cdot} = \sum_{t=1}^{r} p_{it}, \quad p_{\cdot i} = \sum_{s=1}^{r} p_{si}.$$

This model states that the row marginal distribution is equal to the column marginal distribution.

Let F_i^X and F_i^Y denote the cumulative marginal probabilities of X and Y, respectively; those are

$$F_i^X = \Pr(X \le i) = \sum_{k=1}^{i} p_{k\cdot}, \quad F_i^Y = \Pr(Y \le i) = \sum_{k=1}^{i} p_{\cdot k} \quad (i = 1, \ldots, r-1).$$

Using the cumulative marginal probabilities $\{F_i^X, F_i^Y\}$, the MH model can be expressed as

$$F_i^X = F_i^Y \quad (i = 1, \ldots, r-1).$$

This states that the row cumulative marginal distribution is identical with the column cumulative marginal distribution.

When the MH model does not hold, we are interested in measuring the degree of departure from MH. For square contingency tables with ordered categories, Tahata, Iwashita and Tomizawa (2006) proposed a measure based on the cumulative marginal probabilities (denoted by Ω in Section 2) which represents the degree of departure from MH. The measure Ω has characteristics that (i) Ω lies between 0 and 1, (ii) $\Omega = 0$ if and only if the MH model holds, and (iii) $\Omega = 1$ if and only if the degree of departure from MH is the maximum (say, maximum marginal inhomogeneity), in the sense that $F_i^X = 0$ (then $F_i^Y \ne 0$) for all $i = 1, \ldots, r-1$ (say, maximum lower-marginal inhomogeneity) or $F_i^Y = 0$ (then $F_i^X \ne 0$) for all $i = 1, \ldots, r-1$ (say, maximum right-marginal inhomogeneity). For example, the measure Ω takes the maximum value 1 in the sense of maximum lower-marginal inhomogeneity and maximum right-marginal inhomogeneity for the cases in Tables 1a and 1b, respectively. The measure Ω can judge whether or not the MH model holds. However, the measure Ω cannot distinguish the directionality for two types of maximum marginal inhomogeneities (i.e., maximum lower-marginal inhomogeneity or maximum right-marginal inhomogeneity) because of property (iii).

Table 1. The examples of artificial probabilities with the maximum departure from MH.

	(1)	(2)	(3)	(4)	Total		(1)	(2)	(3)	(4)	Total
(a) Lower-marginal inhomogeneity						(c) Upper-marginal inhomogeneity					
(1)	0	0	0	0	0	(1)	0.4	0.3	0.2	0.1	1
(2)	0	0	0	0	0	(2)	0	0	0	0	0
(3)	0	0	0	0	0	(3)	0	0	0	0	0
(4)	0.4	0.3	0.2	0.1	1	(4)	0	0	0	0	0
Total	0.4	0.3	0.2	0.1	1	Total	0.4	0.3	0.2	0.1	1
(b) Right-marginal inhomogeneity						(d) Left-marginal inhomogeneity					
(1)	0	0	0	0.4	0.4	(1)	0.4	0	0	0	0.4
(2)	0	0	0	0.3	0.3	(2)	0.3	0	0	0	0.3
(3)	0	0	0	0.2	0.2	(3)	0.2	0	0	0	0.2
(4)	0	0	0	0.1	0.1	(4)	0.1	0	0	0	0.1
Total	0	0	0	1	1	Total	1	0	0	0	1

Tahata, Kawasaki and Tomizawa (2012) proposed a measure (denoted by Ψ in Section 2) which can distinguish the directionality for two types of maximum marginal inhomogeneities. The measure Ψ has characteristics that (i) Ψ lies between -1 and 1, (ii) $\Psi = -1$ if and only if there is a structure of maximum right-marginal inhomogeneity, (iii) $\Psi = 1$ if and only if there is a structure of maximum lower-marginal inhomogeneity, and (iv) if the MH model holds then $\Psi = 0$, but the converse does not hold. The measure Ψ cannot directly measure the degree of departure from MH because of property (iv) (although the measure Ψ can distinguish the directionality for two types of maximum marginal inhomogeneities).

In a similar manner to the measure vector $\boldsymbol{\Phi} = (\gamma, \phi)^t$ to represent the degree of departure from the S model, we are interested in the measure vector which can judge whether or not the MH model holds and distinguish the directionality for two types of maximum marginal inhomogeneities simultaneously.

The purpose of this paper is to propose the measure vector which represents the degree of departure from MH including directionality for two types of maximum marginal inhomogeneities. The proposed measure vector would be useful for comparing visually degrees of departure from MH in several tables.

2 Measure Vector

Let

$$F^*_{1(i)} = \frac{F^X_i}{\Delta}, \ F^*_{2(i)} = \frac{F^Y_i}{\Delta}, \ Q^*_i = \frac{1}{2}(F^*_{1(i)} + F^*_{2(i)}) \quad (i = 1, \ldots, r-1),$$

and

$$\Delta = \sum_{i=1}^{r-1}(F^X_i + F^Y_i).$$

Assume that $\{F^X_i + F^Y_i \neq 0\}$, we propose a measure vector which represents the degree of departure from MH defined by

$$\boldsymbol{\Lambda} = \begin{pmatrix} \Omega \\ \Psi \end{pmatrix}; \text{ the } 2 \times 1 \text{ vector,}$$

where

$$\Omega = \frac{1}{\log 2} I_{MH},$$

$$\Psi = \frac{4}{\pi}\sum_{i=1}^{r-1}(F^*_{1(i)} + F^*_{2(i)})\left(\theta_i - \frac{\pi}{4}\right),$$

with

$$I_{MH} = \sum_{i=1}^{r-1}\left[F^*_{1(i)}\log\left(\frac{F^*_{1(i)}}{Q^*_i}\right) + F^*_{2(i)}\log\left(\frac{F^*_{2(i)}}{Q^*_i}\right)\right],$$

$$\theta_i = \sin^{-1}\left(\frac{F^Y_i}{\sqrt{(F^X_i)^2 + (F^Y_i)^2}}\right).$$

Note that I_{MH} is the Kullback-Leibler information between $\{F^*_{1(i)}, F^*_{2(i)}\}$ and $\{Q^*_i, Q^*_i\}$. Measures Ω and Ψ are given by Tahata et al. (2006) and Tahata et al. (2012), respectively.

From characteristics of measures Ω and Ψ, we obtain the following theorem.

Theorem 1.

(i) $\boldsymbol{\Lambda} = (0,0)^t$ if and only if the MH model holds,

(ii) $\boldsymbol{\Lambda} = (1,-1)^t$ if and only if there is a structure of maximum right-marginal inhomogeneity,

(iii) $\boldsymbol{\Lambda} = (1,1)^t$ if and only if there is a structure of maximum lower-marginal inhomogeneity.

Therefore, we see that the measure vector $\boldsymbol{\Lambda}$ can measure the degree of departure from MH which includes directionality for two types of maximum marginal inhomogeneities.

3 Approximate Confidence Region for Measure Vector

Let n_{ij} denote the observed frequency in the ith row and jth column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$). Assume that a multinomial distribution applies to the $r \times r$ table. The sample version of Ω, i.e., $\widehat{\Omega}$, is given by Ω with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \sum\sum n_{ij}$. Similarly, the sample version of Ψ, i.e., $\widehat{\Psi}$, is given by Ψ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Therefore, the sample version of $\boldsymbol{\Lambda}$, i.e., $\widehat{\boldsymbol{\Lambda}}$, is given by $\boldsymbol{\Lambda}$ with Ω and Ψ replaced by $\widehat{\Omega}$ and $\widehat{\Psi}$, respectively. Using the delta method (see, e.g., Agresti, 2013, p.590), we obtain the following theorem.

Theorem 2.

$\sqrt{n}(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})$ has asymptotically (as $n \to \infty$) a bivariate normal distribution with mean zero vector and the covariance matrix $\boldsymbol{\Sigma}[\widehat{\boldsymbol{\Lambda}}]$, where

$$\boldsymbol{\Sigma}[\widehat{\boldsymbol{\Lambda}}] = \sum_{s=1}^{r} \sum_{t=1}^{r} \begin{pmatrix} p_{st}(w_{st})^2 & p_{st} w_{st} a_{st} \\ p_{st} w_{st} a_{st} & p_{st}(a_{st})^2 \end{pmatrix},$$

with

$$w_{st} = \frac{1}{\Delta \log 2} \left[\sum_{i=1}^{r-1} \left\{ I(s \le i) \log \left(\frac{F_i^X}{F_i^X + F_i^Y} \right) + I(t \le i) \log \left(\frac{F_i^Y}{F_i^X + F_i^Y} \right) \right\} \right.$$
$$\left. - (2r - (s+t))(\log 2)(\Omega - 1) \right],$$

$$a_{st} = \frac{4}{\pi \Delta} \sum_{i=1}^{r-1} \left[\left\{ I(s \le i) + I(t \le i) \right\} \theta_i + \frac{F_i^X + F_i^Y}{(F_i^X)^2 + (F_i^Y)^2} \left\{ -I(s \le i) F_i^Y + I(t \le i) F_i^X \right\} \right]$$
$$- \frac{\left\{ 2r - (s+t) \right\}(\Psi + 1)}{\Delta},$$

and $I(\cdot)$ is the indicator function, $I(\cdot) = 1$ if true, 0 if not.

Let $\widehat{\boldsymbol{\Sigma}[\boldsymbol{\Lambda}]}$ denote $\boldsymbol{\Sigma}[\widehat{\boldsymbol{\Lambda}}]$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. From Theorem 2, an approximate $100(1-p)$ percent confidence region for $\boldsymbol{\Lambda}$ is expressed as

$$n(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^t (\widehat{\boldsymbol{\Sigma}[\boldsymbol{\Lambda}]})^{-1} (\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda}) \le \chi^2_{(1-p;2)},$$

where $\chi^2_{(1-p;2)}$ is the $1-p$ quantile of the chi-square distribution with two degrees of freedom.

Consider comparison between degrees of departure from MH in Tables A and B (with sample sizes n_A and n_B, respectively). For Tables A and B, the measure vectors $\boldsymbol{\Lambda}$ are denoted by $\boldsymbol{\Lambda}_A$ and $\boldsymbol{\Lambda}_B$, respectively and its sample version are denoted by $\widehat{\boldsymbol{\Lambda}}_A$ and $\widehat{\boldsymbol{\Lambda}}_B$, respectively. When n_A and n_B are large, $\widehat{\boldsymbol{\Lambda}}_A - \widehat{\boldsymbol{\Lambda}}_B - (\boldsymbol{\Lambda}_A - \boldsymbol{\Lambda}_B)$ has approximately a bivariate normal distribution with mean zero vector and covariance matrix $\boldsymbol{V}$, where

$$\boldsymbol{V} = \frac{1}{n_A} \boldsymbol{\Sigma}[\widehat{\boldsymbol{\Lambda}}_A] + \frac{1}{n_B} \boldsymbol{\Sigma}[\widehat{\boldsymbol{\Lambda}}_B].$$

Therefore, an approximate $100(1-p)$ percent confidence region for $\boldsymbol{\Lambda}_A - \boldsymbol{\Lambda}_B$ is expressed as

$$(\widehat{\boldsymbol{\Lambda}}_A - \widehat{\boldsymbol{\Lambda}}_B - (\boldsymbol{\Lambda}_A - \boldsymbol{\Lambda}_B))^t \widehat{\boldsymbol{V}}^{-1} (\widehat{\boldsymbol{\Lambda}}_A - \widehat{\boldsymbol{\Lambda}}_B - (\boldsymbol{\Lambda}_A - \boldsymbol{\Lambda}_B)) \le \chi^2_{(1-p;2)},$$

where $\widehat{\boldsymbol{V}}$ is given by $\boldsymbol{V}$ with $\boldsymbol{\Sigma}[\widehat{\boldsymbol{\Lambda}}_A]$ and $\boldsymbol{\Sigma}[\widehat{\boldsymbol{\Lambda}}_B]$ replaced by $\widehat{\boldsymbol{\Sigma}[\boldsymbol{\Lambda}_A]}$ and $\widehat{\boldsymbol{\Sigma}[\boldsymbol{\Lambda}_B]}$, respectively.

4 Examples

4.1 Example 1

Consider the data in Table 2 taken from Tominaga (1979, p.131). These data describe the cross-classification of father's and son's occupational status categories in Japan which were examined in 1955 and 1975. Note that status (1) is Professional and Managers, (2) is Clerical and Sales, (3) is Skilled manual and Semiskilled manual and (4) is Unskilled manual and Farmers. Thus, for the data in Tables 2a and 2b, the structure of lower-marginal inhomogeneity shows that son's status would be higher than father's, and the structure of right-marginal inhomogeneity shows that son's status would be lower than father's.

Table 2. Occupational status for Japanese father-son pair's; from Tominaga (1979, p.131). Note that status (1) is Professional and Managers, (2) is Clerical and Sales, (3) is Skilled manual and Semiskilled manual and (4) is Unskilled manual and Farmers.

Father's status	Son's status				Total
	(1)	(2)	(3)	(4)	
(a) Examined in 1955					
(1)	80	72	37	19	208
(2)	44	155	61	31	291
(3)	26	73	218	45	362
(4)	69	156	166	614	1005
Total	219	456	482	709	1866
(b) Examined in 1975					
(1)	127	101	54	12	294
(2)	86	207	125	13	431
(3)	78	124	310	24	536
(4)	109	206	437	325	1077
Total	400	638	926	374	2338

First, we shall measure degrees of departure from MH in Tables 2a and 2b using the confidence region for $\boldsymbol{\Lambda}$. For Tables 2a and 2b, estimates of $\boldsymbol{\Lambda}$ are

$$\widehat{\boldsymbol{\Lambda}}_{2a} = \begin{pmatrix} 0.014 \\ 0.169 \end{pmatrix} \quad \text{and} \quad \widehat{\boldsymbol{\Lambda}}_{2b} = \begin{pmatrix} 0.029 \\ 0.248 \end{pmatrix},$$

respectively, and estimates of $\boldsymbol{\Sigma}$ are

$$\widehat{\boldsymbol{\Sigma}[\boldsymbol{\Lambda}_{2a}]} = \begin{pmatrix} 0.010 & 0.070 \\ 0.070 & 0.503 \end{pmatrix} \quad \text{and} \quad \widehat{\boldsymbol{\Sigma}[\boldsymbol{\Lambda}_{2b}]} = \begin{pmatrix} 0.018 & 0.081 \\ 0.081 & 0.365 \end{pmatrix},$$
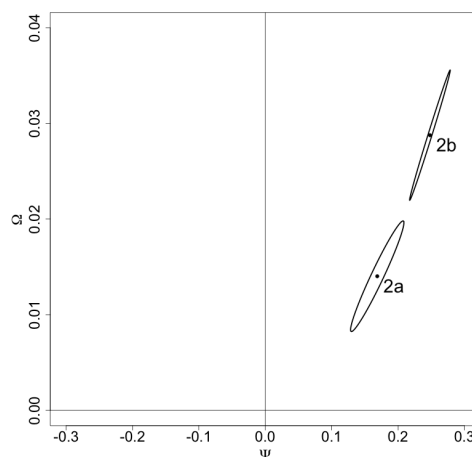
respectively.

Figure 1. Approximate 95% confidence regions for $\boldsymbol{\Lambda}$, applied to Tables 2a and 2b.

Confidence regions do not overlap for Tables 2a and 2b (see Figure 1). From Figure 1, we can see that the data in both Tables 2a and 2b tend to have a structure of lower-marginal inhomogeneity rather than the structure of right-marginal inhomogeneity. Thus, we see that son's status would be higher than father's in both 1955 and 1975.

Next, we shall compare between degrees of departure from MH in Tables 2a and 2b using the confidence region for $\Lambda_{2a} - \Lambda_{2b}$. Note that, for these data, (i) if the confidence region for $\Lambda_{2a} - \Lambda_{2b}$ is included within the first quadrant in Figure 2, we can see that the degree of departure from MH in Table 2a is greater than that in Table 2b, and degree of rise of son's status in 1955 is higher than that in 1975, (ii) if that is included within the second quadrant, we can see that the degree of departure from MH in Table 2a is greater than that in Table 2b, and degree of rise of son's status in 1975 is higher than that in 1955, (iii) if that is included within the third quadrant, we can see that the degree of departure from MH in Table 2b is greater than that in Table 2a, and degree of rise of son's status in 1975 is higher than that in 1955, and (iv) if that is included within the fourth quadrant, we can see that the degree of departure from MH in Table 2b is greater than that in Table 2a, and degree of rise of so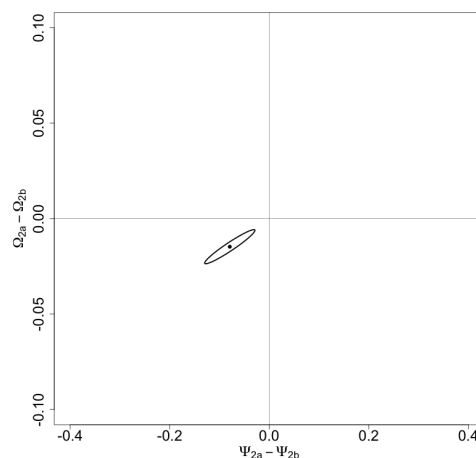n's status in 1955 is higher than that in 1975. The confidence region for $\Lambda_{2a} - \Lambda_{2b}$ is included within the third quadrant (see Figure 2). Thus, we see that the degree of departure from MH in Table 2b is greater than that in Table 2a, and degree of rise of son's status in 1975 is higher than that in 1955.

Figure 2. Approximate 95% confidence region for $\Lambda_{2a} - \Lambda_{2b}$.

4.2   Example 2

Consider the artificial data in Tables 3a, 3b and 3c to show that the two-dimensional measure vector is useful for comparing visually the degree of departure from MH. From Tables 4a, 4b and 4c, we see that all of the value $\widehat{\Omega}$ for Tables 3a, 3b and 3c are equal, although the data in Tables 3a, 3b and 3c have a different structure of marginal inhomogeneity. Also, the value $\widehat{\Psi}$ for Table 3c nearly equals to zero, although the data in Table 3c do not have a structure of marginal homogeneity. From Figure 3, we see the degree of departure from MH, while distinguishing two types of maximum marginal inhomogeneities. Thus, we see that (i) the data in Table 3a have a structure of lower-marginal inhomogeneity, (ii) the data in Table 3b have a structure of right-marginal inhomogeneity, and (iii) the data in Table 3c would have neither the structure of lower-marginal inhomogeneity nor that of right-marginal inhomogeneity.

Table 3. Artificial data (n is sample size).

(a) $n = 740$				(b) $n = 740$				(c) $n = 740$			
10	10	10	10	10	300	10	10	10	10	10	10
300	10	10	10	10	10	10	10	300	10	10	10
10	10	10	10	10	10	10	300	10	10	10	300
10	10	300	10	10	10	10	10	10	10	10	10

Table 4. Estimates of Ω and Ψ, approximate standard errors for $\widehat{\Omega}$ and $\widehat{\Psi}$, and approximate 95% confidence intervals for Ω and Ψ, applied to Tables 3a, 3b and 3c.

	Estimated measure	Standard error	Confidence interval
(a) For Table 3a			
Ω	0.1092	0.0083	(0.0929, 0.1255)
Ψ	0.3038	0.0136	(0.2771, 0.3304)
(b) For Table 3b			
Ω	0.1092	0.0083	(0.0929, 0.1255)
Ψ	-0.3038	0.0136	$(-0.3304, -0.2771)$
(c) For Table 3c			
Ω	0.1092	0.0077	(0.0941, 0.1243)
Ψ	-0.0216	0.0160	$(-0.0529, 0.0097)$

5   Discussions

The proposed measure vector $\boldsymbol{\Lambda}$ depends on the information about category ordering since $\boldsymbol{\Lambda}$ is based on the cumulative marginal probabilities. Now, we shall consider to reverse order of categories.

Let S_i^X and S_i^Y denote the reverse cumulative marginal probabilities of X and Y, respectively, defined by

$$S_i^X = \Pr(X \geq i+1) = \sum_{k=i+1}^{r} p_{k\cdot}, \quad S_i^Y = \Pr(Y \geq i+1) = \sum_{k=i+1}^{r} p_{\cdot k} \quad (i = 1, \ldots, r-1).$$

These are the cumulative marginal probabilities which are taken in reverse order of categories; thus

$$S_i^X = 1 - F_i^X, \quad S_i^Y = 1 - F_i^Y \quad (i = 1, \ldots, r-1).$$

Using the reverse cumulative marginal probabilities $\{S_i^X, S_i^Y\}$, the MH model may further be expressed as

$$S_i^X = S_i^Y \quad (i = 1, \ldots, r-1).$$

Let

$$S_{1(i)}^* = \frac{S_i^X}{\widetilde{\Delta}}, \ S_{2(i)}^* = \frac{S_i^Y}{\widetilde{\Delta}}, \ \widetilde{Q}_i^* = \frac{1}{2}(S_{1(i)}^* + S_{2(i)}^*) \quad (i = 1, \ldots, r-1),$$

where

$$\widetilde{\Delta} = \sum_{i=1}^{r-1}(S_i^X + S_i^Y).$$

Tahata et al. (2006) proposed a measure based on the reverse cumulative marginal probabilities (denoted by $\widetilde{\Omega}$) which represents the degree of departure from MH besides the measure Ω. The $\widetilde{\Omega}$ is defined by Ω with $\{F_{1(i)}^*\}$, $\{F_{2(i)}^*\}$, and $\{Q_i^*\}$ replaced by $\{S_{1(i)}^*\}$, $\{S_{2(i)}^*\}$, and $\{\widetilde{Q}_i^*\}$, respectively. Thus, the characteristics of $\widetilde{\Omega}$ is similar to Ω. However, the definition about the maximum of the degree of departure from MH is $S_i^X = 0$ (then $S_i^Y \neq 0$) for all $i = 1, \cdots, r-1$ (say, maximum upper-marginal inhomogeneity)
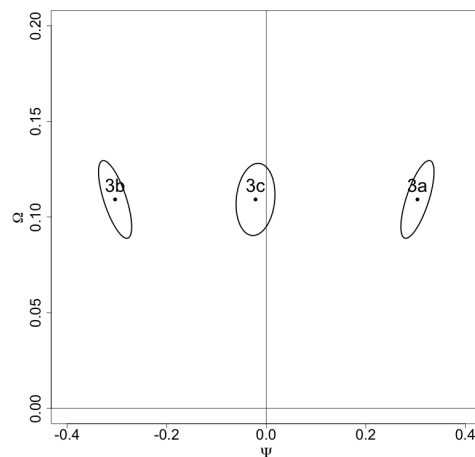
Figure 3. Approximate 95% confidence regions for Λ, applied to Tables 3a, 3b and 3c.

or $S_i^Y = 0$ (then $S_i^X \neq 0$) for all $i = 1, \cdots, r-1$ (say, maximum left-marginal inhomogeneity). For example, the measure $\widetilde{\Omega}$ takes the maximum value 1 in the sense of maximum upper-marginal inhomogeneity and maximum left-marginal inhomogeneity for the cases in Tables 1c and 1d, respectively.

Tahata et al. (2012) proposed a measure based on the reverse cumulative marginal probabilities (denoted by $\widetilde{\Psi}$) which can distinguish above two types of maximum marginal inhomogeneities (namely, maximum upper-marginal inhomogeneity or maximum left-marginal inhomogeneity) besides the measure Ψ. The $\widetilde{\Psi}$ is defined by Ψ with $\{F_{1(i)}^*\}$ and $\{F_{2(i)}^*\}$ replaced by $\{S_{1(i)}^*\}$ and $\{S_{2(i)}^*\}$, respectively. Therefore, two-dimensional measure vector $\widetilde{\Lambda}$ with above measures $\widetilde{\Omega}$ and $\widetilde{\Psi}$ as the elements can simultaneously analyze the degree of departure from MH and directionality for alternative two types of maximum marginal inhomogeneities (i.e., maximum upper-marginal inhomogeneity or maximum left-marginal inhomogeneity).

Also, because of the definition of the reverse cumulative marginal probabilities, we see that the measure vector Λ can measure the directionality for alternative two types of maximum marginal inhomogeneities (i.e., maximum upper-marginal inhomogeneity or maximum left-marginal inhomogeneity) if we apply reverse order for categories in the table.

6   Concluding Remarks

For the analysis of square contingency tables with ordered categories, the proposed measure vector can represent the degree of departure from the MH model including its directionality for two types of maximum marginal inhomogeneities (i.e., maximum lower-marginal inhomogeneity or maximum right-marginal inhomogeneity). And, it is useful for comparing visually the degree of departure from MH in several tables.

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