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Asymptotic Stability for the Initial-Boundary Value Problem of a Semi-linear Wave Equation with Damping

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Abstract In this paper, we study the stability of the solutions to the initial boundary value problem of a semi-linear wave equation with damping $v_{tt}+v_t+f(v_x)=v_{xx}$, on a half line $\mathbb{R}_+$. We show that the solution to the initial-boundary value problem exists as a whole provided that initial datas $\|(v_0-\bar{v})(x)\|_2+\|v_1(x)\|_1$ and the strength of wave $\delta=|v_+-v_-|$ are sufficiently small, In addtion, when the time is sufficiently large, the solution converges to the diffuse wave $\bar{v}(\frac{x}{\sqrt{1+t}})$ at a certain speed, where $\bar{v}(\frac{x}{\sqrt{1+t}})$ is a self-similar solution of one dimension equation $\bar{v}_t+C_0\bar{v}_x^2=\bar{v}_{xx}, v(\pm\infty,t)=v_\pm, v_+\neq v_-$,with $C_0=\frac{1}{2}\frac{\partial^2 f(\xi)}{\partial\xi_1^2}\Big|_{\xi=0}$.

Keywords: semi-linear wave equation, nonlinear diffusion wave, initial-boundary value problem, convergence rate, energy estimate.

1 Introduction

In this paper,we consider the initial-boundary value problem of a semi-linear wave equation with damping

$$v_{tt}+v_t+f(v_x)=v_{xx}, \tag{1.1}$$

with the initial datas

$$v(x,0)=v_0(x),\quad v_t(x,0)=v_1(x), \tag{1.2}$$

where $(x,t)\in\mathbb{R}_+\times\mathbb{R}_+$; f is a given smooth function with $f(0)=f'(0)=0$; and the initial datas have limits at infinity, that is,

$$v\,(0,0)=v_-,\quad v\,(+\infty,0)=v_+,\quad v_1(0,0)=v_1(+\infty,0)=0.$$

where $v_\pm$ are given unequal positive constants, and the boundary condition

$$v\,(0,t)=v_- \tag{1.3}$$

In this paper, we concern about the asymptotic stability of diffusion wave of the initial-boundary value problem of a semi-linear wave equation with damping. From Darcy's law and asymptotic analyses, it is well known that the left of (1.1) decays faster than the right. Generally speak, the corresponding solution shows some decay properties in large time behavior, when an evolution equation has a damping term. Especially when the end states is $v_-\neq v_+$, it is usually accompanied by this wave phenomenon. In other words, the solution of the evolution equation may tends to a diffusion wave as the time $t\to+\infty$. In fact, many authors have confirmed this conclusion. For the Cauchy problem, In [7], Hsiao and Liu firstly proved that the asymptotic behavior of the solution of a hyperbolic conservation equation with damping on the nonlinear diffusion wave under some smallness conditions; Nishihara [6] obtained a better convergence rate in L^2 and L^∞ norm about the same problem. Zhao showed that for a certain class of given large initial data, the p-system with frictional damping admitted a unique global smooth solution and such a solution tended time-asymptotically in [5], at the $L^p(2\leq p\leq\infty)$ decay rates to the corresponding nonlinear diffusion wave. For n-dimension case, Huang, Mei and Wang [4] studied the n-dimensional bipolar hydrodynamic model for semiconductors in the form of Euler-Poisson equations and proved the stability of the nonlinear diffusion wave for this model. For other related results about Cauchy problem, one can refer to [1,2,3,10,13,15] and some references therein.

For the initial-boundary value problem on a half line R_+, Nishihara and Yang [12] considered the large time behavior of the solution of the p-system with linear damping respectively under Dirichlet boundary condition ($u(0,t)=0$) and the Neumann boundary condition ($u_x(0,t)=0$). Precisely, in the case of null-Dirichlet boundary condition, They obtained the optimal convergence rate by using the Green function of the diffusion equation with constant coefficients, and proved the solution (v,u) tend to $(v_+,0)$ as t tends to infinity. In the case of null-Neumann boundary condition, they proved the solution $v(x,t)$ tends to diffusion wave $\bar{v}(x,t)$, and obtained the optimal convergence rate when $v_0(0)=v_+$. For the asymptotic behavior of solutions to the other equations with nonlinear damping, we refer to [8],[9],[11],[12] and some references therein.

In light on equation (1.1), We have already studied the well-posedness and large time behavior of the solution of the Cauchy problem. In this paper, we further guess that there is a wave phenomenon for the initial-boundary value problem. Precisely, We are interested in the global solutions in time of the initial-boundary value problem of the semi-linear wave equation (1.1)∼(1.3) and the asymptotic stability to diffusion wave $\bar{v}(\frac{x}{\sqrt{1+t}})$, as t tends to $+\infty$, where $\bar{v}(\frac{x}{\sqrt{1+t}})$ is a self-similar solution of the one-dimensional quaslinear parabolic equation

$$\begin{cases} \bar{v}_t + C_0 \bar{v}_{x_1}^2 = \bar{v}_{x_1 x_1}, \\ \bar{v}(x_1,t) \to v_\pm, \text{as } x \to \pm\infty, \end{cases} \tag{1.4}$$

where $C_0 = \frac{1}{2}\frac{d^2 f(\xi)}{d\xi^2}\Big|_{\xi=0}$.

Remark 1.1 *By the Hopf-Cole transformation $\bar{v} = -\frac{1}{C_0}\ln u$, (1.4) is equivalent to heat equation*

$$u_t = u_{x_1 x_1}, \tag{1.5}$$

which has a self-similar solution $u(\frac{x_1}{\sqrt{1+t}})$.

In [7], some fundamental dissipative properties of $u(\frac{x}{\sqrt{1+t}})$ have been given clearly. In fact, by the above Hopf-cole transformation, One can easily know that $\bar{v}(\frac{x}{\sqrt{1+t}})$ has the same decay properties which play a role in the process of proof, So we present it in the Lemma below:

Lemma 1.1 *For $2 \le p < +\infty$ and positive constant σ depending on v_+ and v_-, $\bar{v}(\frac{x}{\sqrt{1+t}})$ satisfies that*

$$\left\|\partial_x^k \partial_t^l \bar{v}\right\|_{L^p(\mathbb{R})} = O(1)\left|v_+ - v_-\right|(1+t)^{-\frac{k}{2}-l+\frac{1}{2p}}, \tag{1.6}$$

$$\partial_x^k \partial_t^l \bar{v} = O(1)\left|v_+ - v_-\right|(1+t)^{-\frac{k}{2}-l}\omega(x,t), \tag{1.7}$$

where $k=1,2,...;l=0,1,2$ and $\omega(x,t) = exp\{-\frac{\sigma x^2}{1+t}\}$.

The main purpose of this paper is to show the global existence and nonlinear stability of diffusion wave $\bar{v}(\frac{x}{\sqrt{1+t}})$ for the initial-boundary value problem (1.1)∼(1.3). Concretely, we establish a perturbation equation for $v(x,t) - \bar{v}(\frac{x}{\sqrt{1+t}})$, then do some estimates on the perturbation equation by applying the elementary time-weighted energy estimate. However, it exists a lot of difficulties at the actual calculation due to the boundary effect and time-depending damping. One will find that the important inequality used in [1] does not work in the estimation of the term $\int_0^t \int_{\mathbb{R}_+} (1+t)^{-1}\varphi^2\omega^2$, Fourtunately, we can use the Poincaré inequality. Our main theorem can be stated as follows:

Theorem 1.1 *Suppose that $f(0) = f'(0) = 0$ and $(v_0, v_1) \in H_2 \times H_1$, then there exists a constant $\bar{\delta} > 0$ such that if*

$$\|(v_0 - \bar{v})(x)\|_{H^2(\mathbb{R}_+)} + \|v_1(x)\|_{H^1(\mathbb{R}_+)} + \delta \le \bar{\delta}, \tag{1.8}$$

the initial-boundary value problem (1.1) exists a unique global solution $v(x,t)$ satisfying

$$v(x,t) - \bar{v}(\tfrac{x}{\sqrt{1+t}}) \in C([0,\infty), H^2(\mathbb{R}_+)),$$
$$v_x(x,t) - \bar{v}_x(\tfrac{x}{\sqrt{1+t}}) \in L^2([0,\infty), H^1(\mathbb{R}_+)),$$
$$v_t(x,t) - \bar{v}_t(\tfrac{x}{\sqrt{1+t}}) \in C([0,\infty), H^1(\mathbb{R}_+)) \cap L^2([0,\infty), H^1(\mathbb{R}_+)),$$

moreover, for any $k=1,2$, we have

$$\left\|\partial_x^k v(\cdot,t)-\partial_x^k \bar{v}(\cdot,t)\right\|_{L^2(\mathbb{R}_+)} \leq C\bar{\delta}(1+t)^{-\frac{k}{2}}, \tag{1.9}$$

and

$$\|v(\cdot,t)-\bar{v}(\cdot,t)\|_{L^\infty(\mathbb{R}_+)} \leq C\bar{\delta}(1+t)^{-\frac{1}{4}}, \tag{1.10}$$

where $\delta=|v_+-v_-|$.

Notations and preliminaries: Throughout this paper, we denote some positive constants only depending on the function f by C and $O(1)$ without any confusion. For function spaces, $L^p(\mathbb{R}_+)$ is Lebesgue space of measurable function on $\mathbb{R}_+$ whose p-th powers are integrable, with its norm

$$\|f\|_{L^p(\mathbb{R}_+)}=\left(\int_{\mathbb{R}^n}|f|^p\,dx\right)^{\frac{1}{p}}, 1\leq p<\infty,$$

and the norm simply denote by $\|\cdot\|$ when $p=2$. Denote the usual l-th order Sobolev spaces on $\mathbb{R}_+$ by $H^l(\mathbb{R}_+)$ with its norm

$$\|f\|_l=(\sum_{k=0}^{l}\left\|D_x^k f\right\|^2)^{\frac{1}{2}},$$

In particular, $\|\cdot\|_0=\|\cdot\|_{L^2}=\|\cdot\|$ when $l=0$. Generally, the integral region $\mathbb{R}_+$ will be omitted for concise layout.

2 Priori Estimates

In this section, we proceed to establish some estimates by energy method which prepare for the proof of Theorem 1.

Let $\varphi=v(x,t)-\bar{v}(\frac{x}{\sqrt{1+t}})$, where $\bar{v}(\frac{x}{\sqrt{1+t}})$ is the self-similar solution of (1.4), which approximates the equation (1.1) provided that it satisfies

$$\bar{v}_t+f(\bar{v}_{x_1})=\bar{v}_{x_1x_1}+E, \tag{2.1}$$

where

$$E=: f(\bar{v}_{x_1})-C_0\bar{v}_{x_1}^2=O(1)\left|v_+-v_-\right|^3(1+t)^{-\frac{3}{2}}e^{-\frac{3\sigma x_1^2}{(1+t)}},$$

thus, φ solves the following initial-boundary value problem:

$$\begin{cases}\varphi_{tt}+\varphi_t-\varphi_{xx}=F-\bar{v}_{tt}-E,\\ (\varphi,\varphi_t)(x,0):=(\varphi_0,\varphi_1)(x)=(\hat{v}_0-\bar{v},\hat{v}_1)(x)\\ \varphi\left(x,t\right)|_{x=0}=0.\end{cases} \tag{2.2}$$

where

$$F=-f(\varphi_x+\bar{v}_x)+f(\bar{v}_x),$$

To make the proof of Theorem 1.1 concise, we divide it into the local existence and priori estimate, where the local existence theorem can be proved by standard method, we omit it here. we focus our attention on the priori estimates. For the end, we restrict ourselves to deal with the problem under the following assumption:

Assumption 2.1 *For $t\in[0,T]$, $\varphi\in X(0,T)$ is a solution of (2.2), we assume that*

$$N(t):=\sup_{0\leq t\leq T}\left\{\sum_{k=0}^{2}(1+t)^k\left\|\partial_x^k\varphi\left(\cdot,t\right)\right\|^2+\sum_{k=0}^{1}(1+t)^{k+1}\left\|\partial_x^k\varphi_t\left(\cdot,t\right)\right\|^2\right\}\leq\varepsilon^2. \tag{2.3}$$

where ε is a small positive constant.

Proposition 2.1 *Suppose that $\varphi \in X(x,t)$ is a solution of (2.2),then there are suitable small positive constant δ, it holds that*

$$\|\varphi(t)\|_2^2+\|\varphi_t(t)\|_1^2+\int_0^t\left(\|\varphi_x(\tau)\|_1^2+\|\varphi_t(\tau)\|_1^2\right)\mathrm{d}\tau\le C\|\varphi_0\|_2^2+C\|\varphi_1\|_1^2+C\delta^2, \tag{2.4}$$

where C is independent of T.

To proof proposition 2.1 , we establish a series of estimates of φ .

Lemma 2.1 *Suppose that $\varphi \in X(x,t)$ is a solution of (2.2), then under the condition of Theorem 1.1, there are suitable small positive constant δ, it holds that*

$$\|\varphi(t)\|_1^2+\|\varphi_t(t)\|^2+\int_0^t\|(\varphi_t,\varphi_x)(\tau)\|^2\mathrm{d}\tau\le C\left(\|\varphi_0\|_1^2+\|\varphi_1\|^2\right)+C\delta^2, \tag{2.5}$$

where C is independent of T.

Proof. Multiplying $(2.2)_1$ by φ, and using integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\|\varphi(t)\|^2+\int_0^t\|\varphi_x(\tau)\|^2\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}+}F\varphi\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}+}\bar{v}_{tt}\varphi\mathrm{d}x\mathrm{d}\tau\\&=\frac{1}{2}\|\varphi_0\|^2+\int_0^t\|\varphi_t(\tau)\|^2\mathrm{d}\tau-\int_0^t\int_{\mathbb{R}+}E\varphi\mathrm{d}x\mathrm{d}\tau-\int_{\mathbb{R}+}\varphi\varphi_t\mathrm{d}x\bigg|_0^t\end{aligned} \tag{2.6}$$

Now, we estimate the right terms one by one. It follows by the Cauchy-Schwarz inequality, (2.3) and $E=O(1)\delta^3(1+t)^{-\frac{3}{2}}\omega^3$ that

$$\int_{\mathbb{R}+}\varphi\varphi_t\mathrm{d}x\bigg|_0^t\le\varepsilon\|(\varphi,\varphi_t)(t)\|^2+C_\varepsilon\|(\varphi,\varphi_t)(x,0)\|^2. \tag{2.7}$$

and

$$\int_0^t\int_{\mathbb{R}+}E\varphi+\bar{v}_{tt}\varphi\mathrm{d}x\mathrm{d}\tau\le C\delta\|\varphi\|_{L^\infty}\int_0^t\int_{\mathbb{R}+}(1+t)^{-\frac{3}{2}}\omega\mathrm{d}x\mathrm{d}\tau\le C\delta^2, \tag{2.8}$$

since $\int_0^t\int_{\mathbb{R}}(1+\tau)^{-s}\omega^d dx_1 dt$ is bounded for any $d>0$, and $s>\frac{3}{2}$.

On the other hand, By use Taylor formula and $f(0)=f'(0)=0$, we get

$$\begin{aligned}|F|&=|f(\varphi_x+\bar{v}_x)-f(\bar{v}_x)|=\left|f'(\bar{v}_x)\varphi_x+O(1)|\varphi_x|^2\right|\\&=O(1)\left|\varphi_x^2\right|+O(1)\delta(1+t)^{-\frac{1}{2}}\left|\varphi_x\omega\right|,\end{aligned} \tag{2.9}$$

thus, by using priori assumption, Cauchy-Schwarz inequality and boundary condition (1.3), we have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_+}F\varphi\mathrm{d}x\mathrm{d}\tau&\le C\int_0^t\int_{\mathbb{R}_+}\left(\varphi_x^2|\varphi|+\delta(1+\tau)^{-\frac{1}{2}}|\varphi||\varphi_x\omega|\right)\mathrm{d}x\mathrm{d}\tau\\&\le C(\varepsilon+\delta)\int_0^t\|\varphi_x\|^2\mathrm{d}\tau+C\delta\int_0^t\int_{\mathbb{R}_+}(1+t)^{-1}\varphi^2\omega^2\mathrm{d}x\mathrm{d}\tau\\&\le C(\varepsilon+\delta)\int_0^t\|\varphi_x\|^2\mathrm{d}\tau+C\delta\int_0^t\int_{\mathbb{R}_+}\frac{x}{(1+t)}e^{-\frac{2\sigma x^2}{1+t}}\|\varphi_x\|_{L^2(\mathbb{R})}^2\mathrm{d}x\mathrm{d}\tau\\&\le C(\varepsilon+\delta)\int_0^t\|\varphi_x\|^2\mathrm{d}\tau+C\delta\int_0^t\|\varphi_x\|^2\mathrm{d}\tau\\&\le C(\varepsilon+\delta)\int_0^t\|\varphi_x\|^2\mathrm{d}\tau+C\delta^2.\end{aligned} \tag{2.10}$$

where in the third inequality, the following Poincaré inequality has been used

$$|\varphi(x,t)| \le |\varphi(0,t)| + x^{\frac{1}{2}}\|\varphi_x\|_{L^2(\mathbb{R})}, x \in \mathbb{R}. \tag{2.11}$$

Substituting (2.7), (2.8) and (2.10) into (2.6), we have

$$\|\varphi(t)\|^2 + \int_0^t \|\varphi_x(\tau)\|^2 \mathrm{d}\tau \leqslant C\varepsilon\|\varphi_t(t)\|^2 + \int_0^t \|\varphi_t\|^2 \mathrm{d}\tau + \mathrm{C}\left(\|\varphi_0\|^2 + \|\varphi_1\|^2 + \delta^2\right). \tag{2.12}$$

Next, we estimate $\|\varphi_t\|$. Multiplying $(2.2)_1$ by φ_t, and integrating the result on $[0,t]\times\mathbb{R}_+$, we have

$$\begin{aligned}&\frac{1}{2}\|(\varphi_x,\varphi_t)(t)\|^2 + \int_0^t \|\varphi_t(\tau)\|^2 \mathrm{d}\tau + \int_0^t\int_{\mathbb{R}_+} F\varphi_t \mathrm{d}x\mathrm{d}\tau + \int_0^t\int_{\mathbb{R}_+} \bar{v}_{tt}\varphi_t \mathrm{d}x\mathrm{d}\tau\\ =&\frac{1}{2}\|(\varphi_x,\varphi_t)(x,0)\|^2 - \int_0^t\int_{+} E\varphi_t \mathrm{d}x\mathrm{d}\tau\end{aligned} \tag{2.13}$$

By using (2.9), $E = O(1)\delta^3(1+t)^{-\frac{3}{2}}\omega^3$ and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}_+} F\varphi_t \mathrm{d}x\mathrm{d}\tau\right| \le& C\int_0^t\int_{\mathbb{R}_+}\left(\varphi_x^2|\varphi_t| + \delta(1+t)^{-\frac{1}{2}}|\varphi_t||\varphi_x\omega|\right)\mathrm{d}x\mathrm{d}\tau\\ \le& C(\varepsilon+\delta)\int_0^t \|(\varphi_x,\varphi_t)(\tau)\|^2 \mathrm{d}\tau,\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}_+} E\varphi_t \mathrm{d}x\mathrm{d}\tau\\ \leqslant& C\delta^3\int_0^t\int_{\mathbb{R}_+}(1+t)^{-\frac{3}{2}}\omega^3\varphi_t \mathrm{d}x\mathrm{d}\tau\\ \leqslant& C\delta\int_0^t \|\varphi_t(\tau)\|^2 \mathrm{d}\tau + C\delta^2,\end{aligned} \tag{2.15}$$

Thus, choose suitable small positive constant δ, we have

$$\begin{aligned}&\|(\varphi_x,\varphi_t)(t)\|^2 + \int_0^t \|\varphi_t(\tau)\|^2 \mathrm{d}\tau\\ \le& C(\varepsilon+\delta)\int_0^t \|\varphi_x(\tau)\|^2 \mathrm{d}\tau + \mathrm{C}\|(\varphi_x,\varphi_t)(x,0)\|^2 + C\delta^2.\end{aligned} \tag{2.16}$$

combining (2.12) and (2.16), then Lemma 2.1 have been proved.

Lemma 2.2 *Suppose that $\varphi \in X(x,t)$ is a solution of (2.2), then under the condition of Theorem 1.1, there is a suitable small positive constant δ, such that φ satisfies*

$$\|\varphi_x(t)\|_1^2 + \|\varphi_{xt}(t)\|^2 + \int_0^t \|(\varphi_{xt},\varphi_{xx})(\tau)\|^2 \mathrm{d}\tau \le C\|\varphi_0\|_2^2 + C\|\varphi_1\|_1^2 + C\delta^2, \tag{2.17}$$

where C is independent of T.

Proof. By differentiating both sides of equation $(2.2)_1$ with respect to x, we obtain

$$\varphi_{xtt} + \varphi_{xt} + \bar{v}_{xtt} + F_x = \varphi_{xxx} - E_x, \tag{2.18}$$

First, multiplying (2.18) by φ_x, and integrating the result on $[0,t]\times\mathbb{R}_+$, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}_+}\left(\varphi_x^2+2\varphi_{xt}\varphi_x\right)dx\bigg|_0^t+\int_0^t\int_{\mathbb{R}_+}\varphi_{xx}^2\mathrm{d}x\mathrm{d}\tau\\
=&-\int_0^t\int_{\mathbb{R}_+}\bar{v}_{xtt}\varphi_x\mathrm{d}x\mathrm{d}\tau-\int_0^t\varphi_{xx}\varphi_x\mathrm{d}\tau\bigg|_{x=0}+\int_0^t\int_{\mathbb{R}_+}\left(\varphi_{xx}^2+\varphi_{xt}^2\right)\mathrm{d}\tau\\
&-\int_0^t\int_{\mathbb{R}_+}F_x\varphi_x\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\mathbb{R}_+}E_x\varphi_x\mathrm{d}x\mathrm{d}\tau,
\end{aligned}
\tag{2.19}
$$

By Cauchy-Schwarz inequality, and (2.5), we have

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}F_x\varphi_x\mathrm{d}x\mathrm{d}\tau\\
\leq& C\left(\varepsilon+\delta\right)\left(\int_0^t\int_{\mathbb{R}_+}(1+t)^{-1}|\varphi_x|^2\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}_+}(1+t)^{-\frac{1}{2}}\left|\varphi_{xx}\varphi_x\right|\mathrm{d}x\mathrm{d}\tau\right)\\
\leq& C\left(\varepsilon+\delta\right)\int_0^t\|\varphi_x\|_1^2\mathrm{d}\tau,
\end{aligned}
\tag{2.20}
$$

where in the first inequality,we use the following formula

$$
\begin{aligned}
|F_x|=&\left|\left(f\left(\varphi_x+\bar{v}_x\right)-f\left(\bar{v}_x\right)\right)_x\right|\\
\leq&\left|f'\left(\varphi_x+\bar{v}_x\right)\left(\varphi_{xx}+\bar{v}_{xx}\right)-f'\left(\bar{v}_x\right)\bar{v}_{xx}\right|\\
\leq& C\delta(1+t)^{-1}\left|\varphi_x\right|+C\delta(1+t)^{-\frac{1}{2}}\left|\varphi_{xx}\right|.
\end{aligned}
\tag{2.21}
$$

Since $|E_x|=O\left(1\right)\delta^3(1+t)^{-2}\omega^2$, we have

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}E_x\varphi_x\mathrm{d}x\mathrm{d}\tau\\
\leq& C\delta\int_0^t\int_{\mathbb{R}_+}(1+t)^{-2}\left|\omega^2\right|\left|\varphi_x\right|\mathrm{d}x\mathrm{d}\tau\\
\leq& C\delta\int_0^t\|\varphi_x\|^2\mathrm{d}\tau+C\delta^2.
\end{aligned}
\tag{2.22}
$$

In light of the boundary term, Noting that (1.3), and substituting it into (2.2), we get

$$
\begin{aligned}
\varphi_{xx}|_{x=0}=&\left(f(\varphi_x+\bar{v}_x)-f(\bar{v}_x)+C_0\bar{v}_x^2+\bar{v}_{tt}\right)\big|_{x=0}\\
=&O\left(1\right)\varphi_x{}^2+O\left(1\right)\varphi_x\bar{v}_x+(C_0+O(1))\bar{v}_x^2+\bar{v}_{tt}.
\end{aligned}
\tag{2.23}
$$

On the other hands, according to Sobolev inequality, i.e. for any $t\in[0,T]$, we have

$$
\left|\varphi\left(x,t\right)\right|\leq C\left\|\varphi_x\right\|\left\|\varphi_{xx}\right\|\leq C\|\varphi_x\|_1.
\tag{2.24}
$$

Thus, by using (2.23), (2.24) and (1.7), we have

$$
\begin{aligned}
&\int_0^t\varphi_{xx}\varphi_x\bigg|_{x=0}\mathrm{d}\tau\\
\leq& C\int_0^t\varphi_{xt}\left(\varphi_x{}^2+\varphi_x\bar{v}_x+\bar{v}_x^2+\bar{v}_{tt}\right)\bigg|_{x=0}\mathrm{d}\tau\\
\leqslant& C\left(\varepsilon+\delta\right)\int_0^t\varphi_x\mathrm{d}\tau\bigg|_{x=0}\\
\leqslant& C\left(\varepsilon+\delta\right)\int_0^t\|\varphi_x\|_1^2\mathrm{d}\tau,
\end{aligned}
\tag{2.25}
$$

To sum up, we have

$$\begin{aligned}&\|\varphi_x(t)\|^2 + \int_0^t \|\varphi_{xx}\|^2 \mathrm{d}\tau \\ \leq &C\|\varphi_{xt}(t)\|^2 + C\int_0^t \|\varphi_{xt}\|^2 \mathrm{d}\tau + C\|(\varphi_x, \varphi_{xt})(x,0)\|^2 + C\delta^2,\end{aligned} \tag{2.26}$$

Next, we proceed to estimate $\|\varphi_{xt}(t)\|$. Multiplying (2.18) by φ_{xt}, and integrating the result on $[0,t]\times\mathbb{R}_+$, we have

$$\begin{aligned}&\frac{1}{2}\|(\varphi_{xx}, \varphi_{xt})(t)\|^2 + \int_0^t \|\varphi_{xt}(\tau)\|^2 \mathrm{d}\tau + \int_0^t \int_{+} \bar{v}_{xtt}\varphi_{xt}\mathrm{d}x\mathrm{d}\tau + \int_0^t \int_{+} F_x\varphi_{xt}\mathrm{d}x\mathrm{d}\tau \\ =&\frac{1}{2}\|(\varphi_{xx}, \varphi_{xt})(x,0)\|^2 - \int_0^t \varphi_{xx}\varphi_{xt}\mathrm{d}\tau\Big|_{x=0} - \int_0^t \int_{+} E_x\varphi_{xt}\mathrm{d}x\mathrm{d}\tau\end{aligned} \tag{2.27}$$

We only estimate the boundary term $\int_0^t \varphi_{xx}\varphi_{xt}\mathrm{d}\tau\Big|_{x=0}$. The estimation of other terms are similar to (2.20) and (2.22). By using Integration by parts, (2.23) and Sobolev inequality, we have

$$\begin{aligned}&\int_0^t \varphi_{xx}\varphi_{xt}\Big|_{x=0}\mathrm{d}\tau \\ =&\int_0^t \varphi_{xt}\left(f(\varphi_x+\bar{v}_x) - f(\bar{v}_x) + C_0\bar{v}_x^2 + \bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau \\ \leq &C\int_0^t \varphi_{xt}\left({\varphi_x}^2 + \varphi_x\bar{v}_x + \bar{v}_x^2 + \bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau \\ \leqslant &C\int_0^t \left(\frac{\varphi_x^3}{3}\right)_t + \left(\varphi_x^2\right)_t\bar{v}_x + \varphi_{xt}\bar{v}_x^2 + \varphi_{xt}\bar{v}_{tt}\Big|_{x=0}\mathrm{d}\tau \\ \leqslant &-C\varphi_x\left(\frac{\varphi_x^2}{3} + \varphi_x\bar{v}_x + \bar{v}_x^2 + \bar{v}_{tt}\right)\Big|_{x=0} \\ &+ C\int_0^t \varphi_x\left(\varphi_x\bar{v}_{xt} + 2\bar{v}_x\bar{v}_{xt} + \bar{v}_{ttt}\right)\Big|_{x=0}\mathrm{d}\tau \\ \leqslant &C(\varepsilon+\delta)\left(\|\varphi_x(t)\|_1^2 + \int_0^t \|\varphi_x(\tau)\|_1^2\mathrm{d}\tau\right).\end{aligned} \tag{2.28}$$

Thus, substituting (2.28) into (2.27), and combining (2.26), we finish the proof.

Remark 2.1 *one can deduced Proposition 2.1 immediately provided that ε and δ are small enough by combining Lemma 2.1 and Lemma 2.2.*

Until now, it has been proved that the diffusion wave $\bar{v}(\frac{x}{\sqrt{1+t}})$ is asymptotically stable to the solution $\hat{v}$ of the initial-boundary value problem (2.2), as t tends to infinity. Next we will show the decay rate in time of the solution $\varphi(x,t)$.

Lemma 2.3 *Suppose that $\varphi \in X(x,t)$ is a solution of (2.2), then under the condition of Theorem 1.1, there is a suitable small positive constant δ, such that φ satisfies*

$$(1+t)\|(\varphi_x, \varphi_t)(t)\|^2 + \int_0^t \int_{\mathbb{R}_+} (1+\tau)\varphi_t^2 dx d\tau \leqslant C\left(\|\varphi_0\|_1^2 + \|\varphi_1\|^2\right) + C\delta^2. \tag{2.29}$$

where C is independent of T.

Proof. Multiplying $(2.2)_1$ by $(1+t)\varphi_t$, and using integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}_+} (1+\tau)\left(\varphi_x^2+\varphi_t^2\right)\mathrm{d}x\Big|_0^t + \int_0^t \int_{\mathbb{R}_+} (1+\tau)\varphi_t^2\mathrm{d}x\mathrm{d}\tau = \frac{1}{2}\int_0^t \int_{\mathbb{R}_+} \left(\varphi_x^2+\varphi_t^2\right)\mathrm{d}x\mathrm{d}\tau \\ &-\int_0^t \int_{\mathbb{R}_+} (1+\tau)\bar{v}_{tt}\varphi_t\mathrm{d}x\mathrm{d}\tau - \int_0^t \int_{\mathbb{R}_+} (1+\tau)F\varphi_t\mathrm{d}x\mathrm{d}\tau - \int_0^t \int_{\mathbb{R}_+} (1+\tau)E\varphi_t\mathrm{d}x\mathrm{d}\tau\end{aligned} \tag{2.30}$$

By applying Cauchy-Schwarz inequality and (2.4),we have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,\bar{v}_{tt}\varphi_t\mathrm{d}x\mathrm{d}\tau \leqslant & C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}\left|\varphi_t\omega\right|\mathrm{d}x\mathrm{d}\tau \\ \leqslant & C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,\varphi_t^2\mathrm{d}x\mathrm{d}\tau+C\delta^2.\end{aligned} \tag{2.31}$$

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,F\varphi_t\mathrm{d}x\mathrm{d}\tau \\ \leqslant & C\int_0^t\int_{\mathbb{R}_+}(1+\tau)\left|\varphi_x^2\varphi_t\right|\mathrm{d}x\mathrm{d}\tau+\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{\frac{1}{2}}\left|\varphi_x\omega\varphi_t\right|\mathrm{d}x\mathrm{d}\tau \\ \leqslant & C(\varepsilon+\delta)\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,\varphi_t^2\mathrm{d}x\mathrm{d}\tau+C\delta^2.\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,E\varphi_t\mathrm{d}x\mathrm{d}\tau \leqslant & C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-\frac{1}{2}}\omega^3\varphi_t\mathrm{d}x\mathrm{d}\tau \\ \leqslant & C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,\varphi_t^2\mathrm{d}x\mathrm{d}\tau+C\delta^2.\end{aligned} \tag{2.33}$$

substituting (2.31)∼(2.33) into (2.30), and choosing suitable small ε and δ, we complete the proof.

Lemma 2.4 *(Decay estimates) Suppose that $\varphi\in X(x,t)$ is a solution of (2.2), then under the condition of Theorem 1.1, there exists a suitable small positive constant δ, it holds that*

$$\begin{aligned}&(1+t)^2\|(\varphi_{xx},\varphi_{xt})\,(t)\|^2+\int_0^t(1+\tau)\,\varphi_{xx}^2dx\,d\tau+\int_0^t(1+\tau)^2\varphi_{xt}^2dx\,d\tau \\ \leqslant & C\left\|\varphi_0\right\|_2^2+C\left\|\varphi_1\right\|_1^2+C\delta^2.\end{aligned} \tag{2.34}$$

where C is independent of T.

Proof. Multiplying (2.18) by $(1+t)\varphi_x$, and using integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}_+}(1+\tau)\left(\varphi_x^2+2\varphi_{xt}\varphi_x\right)\mathrm{d}x\Big|_0^t+\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,\varphi_{xx}^2\mathrm{d}x\mathrm{d}\tau \\ =&\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,\varphi_{xt}^2\mathrm{d}x\mathrm{d}\tau+\frac{1}{2}\int_{\mathbb{R}_+}\varphi_x^2\mathrm{d}x\Big|_0^t+\frac{1}{2}\int_0^t\int_{\mathbb{R}_+}\varphi_x^2\mathrm{d}x\mathrm{d}\tau-\int_0^t(1+\tau)\,\varphi_{xx}\varphi_x\mathrm{d}\tau\Big|_{x=0} \\ &-\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,\bar{v}_{tt}\varphi_x\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,F\varphi_x\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,E\varphi_x\mathrm{d}x\mathrm{d}\tau\end{aligned} \tag{2.35}$$

we only estimate the boundary terms similarly, by Sobolev inequality and the priori assumption, we have

$$\begin{aligned}&\int_0^t(1+\tau)\,\varphi_{xx}\varphi_x\Big|_{x=0}\mathrm{d}\tau \\ =&\int_0^t(1+\tau)\,\varphi_x\left(f(\varphi_x+\bar{v}_x)-f\left(\bar{v}_x\right)+C_0\bar{v}_x^2+\bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau \\ =&\int_0^t(1+\tau)\,\varphi_x\Big(O\left(1\right)\left(\varphi_x+\bar{v}_x\right)^2+C_0\bar{v}_x^3+\bar{v}_{tt}\Big)\Big|_{x=0}\mathrm{d}\tau \\ \leqslant & C\left(\varepsilon+\delta\right)\int_0^t\varphi_x\Big|_{x=0}\mathrm{d}\tau \\ \leqslant & C\left(\varepsilon+\delta\right)\int_0^t\left\|\varphi_x\left(\tau\right)\right\|_1^2\mathrm{d}\tau,\end{aligned} \tag{2.36}$$

substituting (2.36) into (2.35), we have

$$
\begin{aligned}
&(1+t)\left\|\varphi_x(t)\right\|^2+\int_0^t(1+\tau)\,\varphi_{xx}^2\mathrm{d}x\mathrm{d}\tau \\
\leqslant& C\left\|\varphi_0\right\|_2^2+C\left\|\varphi_1\right\|_1^2+C\int_0^t(1+\tau)\,\varphi_{xt}^2\mathrm{d}x\mathrm{d}\tau+C\delta^2.
\end{aligned}
\tag{2.37}
$$

Next, Multiplying (2.18) by $(1+t)\varphi_{xt}$, and using integration by parts, we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+}(1+\tau)\left(\varphi_{xt}^2+\varphi_{xx}^2\right)\mathrm{d}x\Big|_0^t+2\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,\varphi_{xt}^2\mathrm{d}x\mathrm{d}\tau \\
=&\int_0^t\int_{\mathbb{R}_+}\left(\varphi_{xx}^2+\varphi_{xt}^2\right)\mathrm{d}x\mathrm{d}\tau-2\int_0^t(1+\tau)\,\varphi_{xx}\varphi_{xt}\mathrm{d}\tau\Big|_{x=0}-2\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,\bar{v}_{tt}\varphi_x\mathrm{d}x\mathrm{d}\tau \\
&-2\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,F\varphi_x\mathrm{d}x\mathrm{d}\tau-2\int_0^t\int_{\mathbb{R}_+}(1+\tau)\,E\varphi_x\mathrm{d}x\mathrm{d}\tau
\end{aligned}
\tag{2.38}
$$

according to boundary condition and Sobolev inequality ,we have

$$
\begin{aligned}
&\int_0^t(1+\tau)\,\varphi_{xx}\varphi_{xt}\Big|_{x=0}\mathrm{d}\tau \\
=&\int_0^t(1+\tau)\,\varphi_{xt}\left(f(\varphi_x+\bar{v}_x)-f(\bar{v}_x)+C_0\bar{v}_x^2+\bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau \\
=&\int_0^t(1+\tau)\,\varphi_{xt}\left(O(1)\,(\varphi_x+\bar{v}_x)^2+C_0\bar{v}_x^2+\bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau \\
=&\int_0^t(1+\tau)\left(O(1)\left(\frac{\varphi_x^3}{3}\right)_t+O(1)\left(\varphi_x^2\right)_t\bar{v}_x+\left(O(1)+C_0\right)\varphi_{xt}\bar{v}_x^2+\varphi_{xt}\bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau \\
=&-(1+\tau)\,\varphi_x\left(O(1)\,\frac{\varphi_x^2}{3}+O(1)\,\varphi_x\bar{v}_x+\left(O(1)+C_0\right)\bar{v}_x^2+\bar{v}_{tt}\right)\Big|_{x=0}\Big|_0^t \\
&+\int_0^t(1+\tau)\,\varphi_x\left(O(1)\,\varphi_x\bar{v}_{xt}+2\left(O(1)+C_0\right)\bar{v}_x\bar{v}_{xt}+\bar{v}_{ttt}\right)\Big|_{x=0}\mathrm{d}\tau \\
&+\int_0^t\varphi_x\left(O(1)\,\varphi_x^2+O(1)\,\varphi_x\bar{v}_x+\left(O(1)+C_0\right)\bar{v}_x^2+\bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau \\
\leqslant& C(\varepsilon+\delta)\,(1+t)^{\frac{1}{2}}\left\|\varphi_x(t)\right\|\left\|\varphi_{xx}(t)\right\|+C(\varepsilon+\delta)\int_0^t\left\|\varphi_x(\tau)\right\|\left\|\varphi_{xx}(\tau)\right\|\mathrm{d}\tau \\
\leqslant& C(\varepsilon+\delta)\,(1+t)\left\|\varphi_{xx}(t)\right\|^2+C(\varepsilon+\delta)\int_0^t\left\|\varphi_x(\tau)\right\|_1^2\mathrm{d}\tau+C\delta^2.
\end{aligned}
\tag{2.39}
$$

substituting (2.39) into (2.38), we have

$$
(1+t)\left\|\left(\varphi_{xx},\varphi_{xt}\right)(t)\right\|^2+\int_0^t(1+\tau)\,\varphi_{xt}^2\mathrm{d}x\mathrm{d}\tau\leqslant C\left(\left\|\varphi_0\right\|_2^2+\left\|\varphi_1\right\|_1^2\right)+C\delta^2.
\tag{2.40}
$$

Combining (2.37) and (2.40), we obtain

$$
\begin{aligned}
&(1+t)\left\|\varphi_x(t)\right\|_1^2+(1+t)\left\|\varphi_{xt}(t)\right\|^2+\int_0^t(1+\tau)\left(\varphi_{xx}^2+\varphi_{xt}^2\right)\mathrm{d}x\mathrm{d}\tau \\
\leqslant& C\left(\left\|\varphi_0\right\|_2^2+\left\|\varphi_1\right\|_1^2\right)+C\delta^2.
\end{aligned}
\tag{2.41}
$$

Finally, multiplying (2.18) by $(1+t)^2\varphi_{xt}$, and using integration by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}_+}(1+\tau)^2\left(\varphi_{xx}^2+\varphi_{xt}^2\right)\mathrm{d}x\Big|_0^t+\int_0^t\int_{\mathbb{R}_+}(1+\tau)^2\varphi_{xt}^2\mathrm{d}x\mathrm{d}\tau\\
=&\int_0^t\int_{\mathbb{R}_+}(1+\tau)\left(\varphi_{xx}^2+\varphi_{xt}^2\right)\mathrm{d}x\mathrm{d}\tau-\int_0^t(1+\tau)^2\varphi_{xx}\varphi_{xt}\mathrm{d}\tau\Big|_{x=0}\\
&-\int_0^t\int_{\mathbb{R}_+}(1+\tau)^2\bar{v}_{tt}\varphi_x\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\mathbb{R}_+}(1+\tau)^2F\varphi_x\mathrm{d}x\mathrm{d}\tau\\
&-\int_0^t\int_{\mathbb{R}_+}(1+\tau)^2E\varphi_x\mathrm{d}x\mathrm{d}\tau
\end{aligned}
\tag{2.42}
$$

By Sobolev inequality and Lemma 1.1, and Poincaré inequality ,we have

$$
\begin{aligned}
&\int_0^t(1+\tau)^2\varphi_{xx}\varphi_{xt}\Big|_{x=0}\mathrm{d}\tau\\
=&\int_0^t(1+\tau)^2\varphi_{xt}\left(f(\varphi_x+\bar{v}_x)-f\left(\bar{v}_x\right)+C_0\bar{v}_x^2+\bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau\\
=&\int_0^t(1+\tau)^2\varphi_{xt}\left((\varphi_x+\bar{v}_x)^2+C_0\bar{v}_x^2+\bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau\\
=&\int_0^t(1+\tau)^2\left(O\left(1\right)\left(\frac{\varphi_x^3}{3}\right)_t+O\left(1\right)\left(\varphi_x^2\right)_t\bar{v}_x+\left(O\left(1\right)+C_0\right)\varphi_{xt}\bar{v}_x^2+\varphi_{xt}\bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau\\
=&-(1+\mathrm{t})^2\varphi_x\left(O\left(1\right)\frac{\varphi_x^2}{3}+O\left(1\right)\varphi_x\bar{v}_x+\left(O\left(1\right)+C_0\right)\bar{v}_x^2+\bar{v}_{tt}\right)\Big|_{x=0}\\
&+\int_0^t(1+\tau)^2\varphi_x\left(O\left(1\right)\varphi_x\bar{v}_{xt}+2\left(O\left(1\right)+C_0\right)\bar{v}_x\bar{v}_{xt}+\bar{v}_{ttt}\right)\Big|_{x=0}\mathrm{d}\tau\\
&+2\int_0^t(1+\tau)\varphi_x\left(O\left(1\right)\varphi_x^2+O\left(1\right)\varphi_x\bar{v}_x+\left(O\left(1\right)+C_0\right)\bar{v}_x^2+\bar{v}_{tt}\right)\Big|_{x=0}\mathrm{d}\tau\\
\leqslant&C\left(\varepsilon+\delta\right)(1+t)\left\|\varphi_x\left(t\right)\right\|\left\|\varphi_{xx}\left(t\right)\right\|+C\left(\varepsilon+\delta\right)\int_0^t\left\|\varphi_x\left(\tau\right)\right\|\left\|\varphi_{xx}\left(\tau\right)\right\|\mathrm{d}\tau\\
\leqslant&C\left(\varepsilon+\delta\right)(1+t)\left\|\varphi_{xx}\left(t\right)\right\|^2+C\left(\varepsilon+\delta\right)\int_0^t\left\|\varphi_x\left(\tau\right)\right\|_1^2\mathrm{d}\tau+C\delta^2.
\end{aligned}
\tag{2.43}
$$

on the other hand,by using Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}(1+\tau)^2F_x\varphi_{xt}\mathrm{d}x\mathrm{d}\tau\\
\leqslant&C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)\left|\varphi_x\varphi_{xt}\right|\mathrm{d}x\mathrm{d}\tau+C\left(\varepsilon+\delta\right)\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{\frac{3}{2}}\left|\varphi_{xx}\varphi_{xt}\right|\mathrm{d}x\mathrm{d}\tau\\
\leqslant&C\left(\varepsilon+\delta\right)\int_0^t\int_{\mathbb{R}_+}(1+\tau)^2\varphi_{xt}^2\mathrm{d}x\mathrm{d}\tau+C\delta\int_0^t\int_{\mathbb{R}_+}(1+\tau)\varphi_{xx}^2\mathrm{d}x\mathrm{d}\tau+C\delta^2\\
\leqslant&C\left(\varepsilon+\delta\right)\int_0^t\int_{\mathbb{R}_+}(1+\tau)^2\varphi_{xt}^2\mathrm{d}x\mathrm{d}\tau+C\delta^2.
\end{aligned}
\tag{2.44}
$$

substituting (2.43) and (2.44) into (2.42), Combining (**??**)and choosing suitable small ε and δ, then we finish the proof.

3 Proof of Theorem 1.1

According to Lemma 2.1 $\sim$ Lemma 2.4, we obtain the uniformity priori estimate provided that

$$\|(v_0-\bar{v})(x)\|_{H^2(\mathbb{R}_+)}+\|v_1(x)\|_{H^1(\mathbb{R}_+)}+\delta\le\bar{\delta}$$

is met. Further, the existence of global solution been guaranteed by applying the local existence theorem and a prior estimation by using continuity argument. Moreover, we obtain the decay rate

$$\sum_{k=0}^{2}(1+t)^{\frac{k}{2}}\left\|\partial_x^k\varphi(\cdot,t)\right\|\leqslant C\delta^2, \tag{3.1}$$

by applying Gargliado-Nirenberg inequality, we have

$$\|\varphi(\cdot,t)\|_{L^\infty}\leqslant C\|\varphi(\cdot,t)\|^{\frac{1}{2}}\|\varphi_{xx}(\cdot,t)\|^{\frac{1}{2}}\leqslant C\delta(1+t)^{-\frac{1}{4}}. \tag{3.2}$$

Thus, Theorem 1.1 is proved.

References

1. C.He, F.M.Huang, and Y.Yong. Stability of planar diffusion wave for nonlinear evolution equation. *Science China Mathematics*, 55(2):337–352, Feb (2012).
2. C.J.Zhu and M.Jiang. L^p-decay rates to nonlinear diffusion waves for p-system with nonlinear damping. *Science in China Series A*, 49(6):721–739, Jun (2006).
3. Donatella Donatelli, Ming Mei, Bruno Rubino, and Rosella Sampalmieri. Asymptotic behavior of solutions to Euler-Poisson equations for bipolar hydrodynamic model of semiconductors. *Journal of Differential Equations*, 255(10):3150–3184, 2013.
4. F.M.Huang, M. Mei, Y.Wang, and T.Yang. Long-time Behavior of Solutions to the Bipolar Hydrodynamic Model of Semiconductors with Boundary Effect. *Siam Journal on Mathematical Analysis*, 44(2):1134–1164, 2012.
5. Hui. J. Zhao. Convergence to Strong Nonlinear Diffusion Waves for Solutions of p-System with Damping. *Journal Of Differential Equations*,2001.
6. K.Nishihara, W.K.Wang, and T.Yang. L^p-convergence rate to nonlinear diffusion waves for p-system with damping. *Journal Of Differential Equations*, 161(1):191–218, FEB 10 2000.
7. L.Hsiao and T.-P.L. Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping. *Communications in Mathematical Physics*, 143(3):599–605, Jan (1992).
8. Lin, C. K. and Mei, M. Asymptotic behavior of solution to nonlinear damped p-system with boundary effect. *Int J Numer Anal Model Ser B*, 1:70–92, 2010.
9. Jiang, M. and Zhu, C. Convergence to strong nonlinear diffusion waves for solutions to p-system with damping on quadrant. *Journal of Differential Equations*, 246(1):50-77, 2009.
10. Ming Mei, Bruno Rubino, and Rosella Sampalmieri. Asymptotic behavior of solutions to the bipolar hydrodynamic model of semiconductors in bounded domain. *Kinetic and Related Models*, 3(3):537–550, SEP 2012.
11. Marcati, P. and Ming, M. and Rubino, B. Optimal Convergence Rates to Diffusion Waves for Solutions of the Hyperbolic Conservation Laws with Damping. *Journal of Mathematical Fluid Mechanics*, 7(2 Supplement):S224–S240, 2005.
12. Nishihara, K. and Tong, Y. Boundary Effect on Asymptotic Behaviour of Solutions to the p-System with Linear Damping. *Journal of Differential Equations*, 156(2):439-458, 1999.
13. S.X.Xie. Asymptotic stability of solutions to the Hamilton-Jacobi equation. *Journal of Mathematical Analysis and Applications*, 470(2):1030–1045, (2019).
14. Wang and Yong and Tan and Zhong. Stability of steady states of the compressible Euler-Poisson system in R-3. *Journal of Mathematical Analysis and Applications*, 2(422):1058–1071, 2015.
15. Y.Yong. Stability of Planar Diffusion Wave for the Quasilinear Wave Equation with Nonlinear Damping. *Acta Mathematicae Applicatae Sinica-English Series*, 31(1):17–30, (2015).

