Solving the Schrödinger Equation with Hartmann Potential by Factorization Method and Supersymmetry

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Abstract In this paper ,we study Schrödinger equation with Hartman potential, and discussed the fundamental concepts of supersymmetric quantum mechanics (SUSYQM) , and factorization method in radial and angular part separately . The energy eigenvalues and (radial) eigenfunctions of the Hartmann potential are subsequently rederived using the techniques of SUSYQM

Keywords: Modified Kratzer potential, Schrödinger equation, supersymmetry approaches, raising and lowering operators.

1 Introduction

In 1972, an exactly solvable ring-shaped potential was introduced by H. Hartmann [3], The Hartmann potential is given by the following expression

$$v(r,\theta) = \chi\sigma^2 \left(\frac{2a_0}{r} - q\chi \frac{a_0^2}{r^2 \sin^2\theta} \right) \epsilon_0 \equiv \frac{\gamma}{r} + \frac{\eta}{r^2 \sin^2\theta} \tag{1}$$

where $a_0 = \frac{\hbar^2}{me^2}$ represents the Bohr radius and $\epsilon_0 = \frac{-me^4}{2\hbar^2}$ is ground state energy of hydrogen atom, and χ, σ, q are three dimensionless parameters.The introduction of the parameter q makes it possible to obtain the Coulomb-like potential by taking $q = 0$ and $\chi\sigma^2 = Z$ in equation(1),[4,8]. This ring-shaped potential was introduced to describe ring-shaped molecules like cyclic polyenes and benzene. Many papers have been devoted to this potential since1972[1,2,5,8].In this article, the authors presents another alternative method of solution in spherical coordinates using supersymmetry and factorizing method.

2 Seperating Variables of the Schrödinger Equation

Now we are going to consider the corresponding potential in spherical coordinates. In order to separate the radial and angular part we consider the following expression,

$$\Psi(r,\theta,\phi) = R(r)H(\theta)\Phi(\varphi) \tag{2}$$

Substituting this equation to the general form of corresponding Schrödinger, we have radial and angular part of equations as following,

$$\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r})R(r) + \frac{2\mu}{\hbar^2} \left[Er^2 - \gamma r - \frac{\hbar^2}{2\mu} n(n+1) \right] R(r) = 0 \tag{3}$$

and

$$\left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta}) - \frac{m'^2}{\sin^2\theta} - \frac{2\mu\eta}{\hbar^2} \frac{1}{\sin^2\theta} + n(n+1) \right] H(\theta) = 0 \tag{4}$$

$$\frac{\partial^2 \Phi}{\partial\varphi^2} + m'^2 \Phi(\varphi) = 0 \tag{5}$$

The solutions in (5) is periodic and must satisfy the period boundary condition

$$\Phi(2\pi+\varphi)=\Phi(\varphi)$$

from which we obtain:

$$\Phi_{m'}(\varphi)=\frac{1}{\sqrt{2\pi}}\exp(\pm im'\varphi),\qquad m'=1,2,...,k \tag{6}$$

Next step, we will try to solve equations (3) and (4) in the following sections.

3 The Solutions of the Radial Part with Factorization Method

In order to solve the radial part of equation (3) we consider the following variables,[17,18]

$$R(r)=u(r)L(r) \tag{7}$$

therefore,

$$rL''(r)+\left(2\frac{u'}{u}r+2\right)L'(r)+\left[2\frac{u''}{u}r+2\frac{u'}{u}+\frac{2\mu}{\hbar^2}Er-\frac{2\mu}{\hbar^2}\gamma-\frac{n(n+1)}{r}\right]L(r)=0 \tag{8}$$

and compare the following associated Laguerre equation [14-16],

$$rL''_{n,m}(r)+(1+\alpha-\beta r)\,L'_{n,m}(r)+\left[\left(n-\frac{m}{2}\right)\beta-\frac{m}{2}\left(\alpha+\frac{m}{2}\right)\frac{1}{r}\right]L_{n,m}(r)=0 \tag{9}$$

the energy spectrum and radial part of wave function are respectively

$$E_{n,m}=\frac{2\mu}{\hbar^2}\frac{\gamma^2}{(\frac{m}{2}-n-\alpha)} \tag{10}$$

and

$$R(r)=r^{\frac{(\alpha-1)}{2}}e^{-\frac{\beta r}{2}}L^{\alpha,\beta}_{n,m}(r). \tag{11}$$

In here we note that the solution associated Laguerre in the Rodrigues representation are,

$$L^{\alpha,\beta}_{n,m}(r)=\frac{a_{n,m}(\alpha,\beta)}{r^{\alpha+\frac{m}{2}}e^{-\beta r}}\left(\frac{d}{dr}\right)^{n-m}(r^{n+\alpha}e^{-\beta r}), \tag{12}$$

where $a_{n,m}(\alpha,\beta)$ is the normalization coefficient, and also obtained by,

$$a_{n,m}(\alpha,\beta)=(-1)^m\sqrt{\frac{\beta^{\alpha+m+1}}{\Gamma(n-m+1)\Gamma(n+\alpha+1)}}. \tag{13}$$

Now we are going to the factorize the second order equation from radial part [14-16]. In that case the first order rasing and lowering operators A_+ and A_- with respect to m are ,

$$\begin{aligned}A_+(m;r)&=\sqrt{r}\frac{d}{dr}-\frac{m-1}{2\sqrt{r}}\\A_-(m;r)&=-\sqrt{r}\frac{d}{dr}-\frac{2\alpha+m-2\beta r}{2\sqrt{r}}\end{aligned} \tag{14}$$

and with respect to n and m are,

$$\begin{aligned}A_+(n,m;r)&=r\frac{d}{dr}-\beta r+\frac{1}{2}(2n+2\alpha-m)\\A_-(n,m;r)&=-r\frac{d}{dr}+\frac{1}{2}(2n-m)\end{aligned} \tag{15}$$

4 The Solutions of the Angular Part of Equation

In order to apply factorization method, we introduce a new variable $x = cos\theta$, so the angular part of equation is,

$$\left(1-x^2\right) H''(x) - \; 2xH'(x) + \left[n(n+1) - \frac{(m'^2 + \frac{2\mu}{\hbar^2}\eta)}{1-x^2}\right] H(x) = 0 \tag{16}$$

We assume the $H(x)$ as follows,

$$H(x) = f(x)p(x) \tag{17}$$

and compare with the following Jacobi equation [13,14],

$$(1-x^2)P''^{(\alpha,\beta)}_{n,m}(x) - [\alpha - \beta + \; (\alpha+\beta+2)x]P'^{(\alpha,\beta)}_{n,m}(x) \tag{18}$$

$$+ \left[n(\alpha+\beta+n+1) - \frac{m(\alpha+\beta+m+(\alpha-\beta)x)}{1-x^2}\right] p^{(\alpha,\beta)}_{n,m}(x) = 0,$$

one can obtain the angular part of wave function,

$$H(x) = \left(\frac{1+x}{1-x}\right)^{(\frac{\beta-\alpha}{4})} (1-x^2)^{\frac{(\alpha+\beta)}{4}} p^{(\alpha,\beta)}_{n,m}(x), \tag{19}$$

and some following results,

$$(\alpha+\beta) = -2m. \tag{20}$$

In that case the first order operators corresponding to the angular part of equation with respect to m are,

$$\begin{aligned} A_+(m;x) &= \sqrt{1-x^2}\frac{d}{dx} + \frac{m-1}{\sqrt{1-x^2}}x \\ A_-(m;x) &= -\sqrt{1-x^2}\frac{d}{dx} + \frac{(\alpha-\beta)+(\alpha+\beta+m)x}{\sqrt{1-x^2}} \end{aligned} \tag{21}$$

with respect to n and m are,

$$\begin{aligned} A(n,m;x) &= \sqrt{(1-x^2)}\frac{d}{dx} + (n-m)x - \frac{(\beta-\alpha)}{2} \\ A^+(n,m;x) &= -\sqrt{(1-x^2)}\frac{d}{dx} + (n+m+1)x - \frac{(\beta-\alpha)}{2} \end{aligned} \tag{22}$$

5 The Supersymmetry Approaches for the Hartmman Potential

In order to discuss the supersymmetry for this model[9-13], we have to consider the ground state wave function. By using the Hamiltonian process and ground state wave function, we can obtain the $V_1(r)$ as following.

$$V_1(r) = \frac{\hbar^2}{2m}\frac{d^2\Psi_0''(r)}{\Psi_0(r)} \tag{23}$$

Solutions of the radial wave equation can be rewritten as

$$H_1\Psi_0(r) = -\frac{\hbar^2}{2m}\frac{d^2\Psi_0}{dr^2} + V_1(r)\Psi_0(r) = 0, \tag{24}$$

Now, we factorize the corresponding Hamiltonian in terms of first order equation , which are called A , A^+

$$H_1 = A^+ A \tag{25}$$

The information from Laguerre equation these first order operators are given by the (rolling) equation .

$$\begin{aligned} A(n,m;r) &= -r\frac{d}{dr} + \frac{1}{2}(2n-m), \\ A^+(n,m;r) &= r\frac{d}{dr} - \beta r + \frac{1}{2}K \end{aligned} \tag{26}$$

where

$$K = (2n + \alpha - m). \tag{27}$$

This leads us to the following potential,

$$V_1(r) = W^2(r) - \frac{\hbar}{\sqrt{2m}} W'(r) \tag{28}$$

This equation is known as Riccit equation ,where $W(r)$ is supper potential and we obtain ,

$$W(r) = -\beta r + \frac{1}{2}K \tag{29}$$

Finally we have ,

$$V_1(r) = \beta^2 r^2 - \beta K r + \frac{1}{4}K^2 + \frac{\hbar}{\sqrt{2m}}\beta \tag{30}$$

Now , we are going to obtain the Hamiltonian H_2 , As $H_2 = AA^+$which is partner of H_1,

$$H_2 = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V_2(r), \tag{31}$$

$$V_2(r) = W^2(r) + \frac{\hbar}{\sqrt{2m}} W'(r)$$

Also the corresponding potential V_2 can be obtained by the following expression ,

$$V_2(r) = \beta^2 r^2 - \beta K r + \frac{1}{4}K^2 - \frac{\hbar}{\sqrt{2m}}\beta \tag{32}$$

where H_2 is ,

$$H_2 = -r^2\frac{d^2}{dr^2} - \left[1 - \frac{1}{2}(2n-m)\right] r\frac{d}{dr} + \left[1 - \frac{1}{2}(2n-m)\right]\beta r + \frac{1}{4}K(2n-m) \tag{33}$$

The potential V_1 and V_2 are supersymmetry partner to each other. On the other hand , we have the matrix supersymmetry for the Hamiltonian H_1 and H_2 ,

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \tag{34}$$

We note here the H_1 and H_2can make closed algebra and they relate to Bosonic and fermionic operators ,

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -r\frac{d}{dr} + \frac{1}{2}(2n-m) & 0 \end{bmatrix} \tag{35}$$

and

$$Q^+ = \begin{bmatrix} 0 & A^+ \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & r\frac{d}{dr} - \beta r + \frac{1}{2}K \\ 0 & 0 \end{bmatrix} \tag{36}$$

Here, we have the following commutation relation ,

$$\left[H, Q\right] = \left[H, Q^{+}\right] = 0 \tag{37}$$

$$\{Q, Q^{+}\} = H, \{Q, Q\} = \{Q^{+}, Q^{+}\} = 0$$

and

$$\left[H, Q\right] = \begin{bmatrix} 0 & 0 \\ H_2 A - A H_1 & 0 \end{bmatrix} \tag{38}$$

In order to satisfy the equations (37)and (38), and

$$H_2 A = A H_1 \tag{39}$$

we have β as follows,

$$\beta = \frac{2n + 2\alpha - 2D - m + 7}{2r} \tag{40}$$

This value of β guarantees the equation(37), $\left[H, Q\right] = 0$

Also from equation(37) $\left[H, Q^{+}\right] = 0$

$$\left[H, Q^{+}\right] = \begin{bmatrix} 0 & H_1 A^{+} - A^{+} H_2 \\ 0 & 0 \end{bmatrix} = 0 \tag{41}$$

$$\Rightarrow H_1 A^{+} = A^{+} H_2$$

By comparing the left and right hand side of above equation we have two conditions, $2n - m = 4$ and $2n - m = 0$, we have,

$$[H, Q^{+}] = 0$$

This completely satisfies the anti-commutation relations , $\{Q, Q^{+}\} = H, \{Q, Q\} = \{Q^{+}, Q^{+}\} = 0$

$$\{Q^{+}, Q^{+}\} = 0 \Rightarrow \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} = 0 \tag{42}$$

$$\{Q, Q\} = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = 0$$

and

$$\{Q, Q^{+}\} = H \Rightarrow \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \tag{43}$$

$$\begin{pmatrix} A^{+}A & 0 \\ 0 & AA^{+} \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = H$$

We note here the super charges commutative together with this happen for degeneracy. Now we are going to continue this process for angular section with aspect of case where the first order operators will be as,

$$A^{+}(n, m; r) = -(1 - r^2)\frac{d}{dr} + (n + m + 1)r + \frac{(\beta - \alpha)}{2} \tag{44}$$

$$A(n, m; r) = (1 - r^2)\frac{d}{dr} + (n - m)r - \frac{(\beta - \alpha)}{2}$$

We take advantage form change of variable $x = \cos\theta$ the first order operators in terms of θwill be as, The corresponding Hamiltonian for H_1 is ,

$$H_1 = -\frac{d^2}{d\theta^2} + n\sin\theta + (\gamma\cos\theta + \rho)\frac{d}{d\theta} + n\gamma\cos^2\theta + m(\alpha - \beta)\cos\theta + -\epsilon\rho \tag{45}$$

the super potential for ground state wave function $W(r) = -\gamma\cos\theta - \rho$ and the potential V_1 will be obtained by the following expression ,

$$V_1(\theta) = (-\gamma\cos\theta - \rho)^2 + \frac{\hbar}{\sqrt{2m}}\gamma\sin\theta = (\gamma\cos\theta)^2 + \rho^2 + 2\gamma\rho\cos\theta - \frac{\hbar}{\sqrt{2m}}\gamma\sin\theta \tag{46}$$

Next step we want to make the partner Hamiltonian $H_2 = AA^+$, the Hamiltonian corresponds to V_2 potential,so we have ,

$$H_2 = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V_2(r), \tag{47}$$

$$V_2(r) = W^2(r) + \frac{\hbar}{\sqrt{2m}}W\prime(r)$$

The final result for the $V_2(r)$ and corresponding Hamiltonian are respectively ,

$$V_2(\theta) = (\gamma\cos\theta)^2 + \rho^2 + 2\gamma\rho\cos\theta + \frac{\hbar}{\sqrt{2m}}\gamma\sin\theta \tag{48}$$

and

$$H_2 = -\frac{d^2}{d\theta^2} - \gamma\sin\theta + (\epsilon - n\cos\theta)\frac{d}{d\theta} + n\gamma\cos^2\theta + (n\rho - \epsilon\gamma)\cos\theta - \epsilon\rho \tag{49}$$

Where $V_1(\theta)$ and $V_2(\theta)$ are partner to each other , the matrix form for Hamiltonian is ,

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \tag{50}$$

This Hamiltonian makes closed algebra where super charges $Q$and Q^+ are respectively ,

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{d}{d\theta} - n\cos\theta + \epsilon & 0 \end{pmatrix} \tag{51}$$

and

$$Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{d}{d\theta} - \gamma\cos\theta - \rho \\ 0 & 0 \end{pmatrix} \tag{52}$$

This also closed form of SL(1,1) algebra , so we have

$$[H, Q] = [H, Q^+] = 0 \tag{53}$$

$$\{Q, Q^+\} = H, \{Q, Q\} = \{Q^+, Q^+\} = 0$$

and

$$[H, Q] = \begin{bmatrix} 0 & 0 \\ H_2A - AH_1 & 0 \end{bmatrix} \tag{54}$$

From equation (53) we have obtained the following condition ,

$$H_2A = AH_1 \tag{55}$$

In that case if the relation $[H, Q]$ wants to be satisfied we will have ,

$$\gamma\cos\theta = -\rho \tag{56}$$

Again we consider the following relation , and

$$[H, Q^+] = \begin{bmatrix} 0 & H_1A^+ - A^+H_2 \\ 0 & 0 \end{bmatrix} = 0 \Rightarrow H_1A^+ = A^+H_2 \tag{57}$$

Also with comparing the left and right hand side of above equation ,we obtain,

$$\begin{aligned} (\rho + \epsilon)\gamma\sin\theta &= 0 \\ \alpha - \beta &= 0 \end{aligned} \tag{58}$$

if we apply the $2n - m = 0$, we have,

$$[H, Q^+] = 0$$

This completely satisfies the following anti-commutation relations , $\{Q, Q^+\} = H, \{Q, Q\} = \{Q^+, Q^+\} = 0$

$$\{Q^+, Q^+\} = 0 \Rightarrow \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} = 0 \tag{59}$$

$$\{Q, Q\} = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = 0$$

and

$$\{Q, Q^+\} = H \Rightarrow \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \tag{60}$$

$$\begin{pmatrix} A^+A & 0 \\ 0 & AA^+ \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = H$$

6 Conclusion

In this paper we introduced the generalized Kratzer potential and wrote the corresponding Schrödinger equation. We connected this equation with Lagure and Jacobi equations and obtained the energy spectrum and wavefunction. Also we factorized the corresponding equation in terms of first order equations. We used these first order equations and discussed the generators algebra. Finally the partner for the Kratzer potential for angular and radial part is obtained by the supersymmetry approches. It may be interesting to continue this research with applying Durbox transformation, in that case one can obtain the corresponding modified potential. We take this potential and obtain the super potential and generators of supersymmetry.

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