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Coherent States of Systems with Non-Equidistant Energy Levels

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Abstract. In the paper we have built and examined the properties of quantum systems with non-equidistant energy levels from a point of view of a new introduced approach – the diagonal operator ordering technique (DOOT). In this frame, we examine also the properties of mixed states described by a canonical density operator. We particularize the obtained results for some particular cases (the system with Hamiltonian whose eigenfunctions are the generalized Laguerre functions, as well as the Pöschl-Teller like potentials, and the infinite quantum well).

Keywords: Coherent states, operator ordering, density operator, energy spectra.

1 Introduction

It is well-known that a most popular and also most applicable quantum model is the one-dimensional harmonic oscillator (HO-1D). An important feature of HO-1D is its equidistant energy levels, which ease the various mathematical characterizations of their different physical properties, especially in the case of mixed (thermal) states. On the other hand, among the quantum systems allowing an exact solution of the nonrelativistic stationary Schrödinger equation, a special place is occupied by the systems with non-equidistant energy levels. Generally, the appearance of non-equidistant energy levels is determined by the anharmonic character of the potential. Such systems are, e.g. the infinitely deep square-well potential, the potential whose eigenfunctions are the generalized Laguerre functions, the Pöschl-Teller like potential, the Morse potential and so on (see, [1], and references therein). In this book were examined the coherent states (CSs) of these potentials in the frame of factorization method. Also, in [2] were built the CSs for systems related to the generalized Laguerre functions. These CSs are known in the quantum optical literature as belonging to the nonlinear coherent states (NCSs) which generally is an overcomplete set of vectors in Hilbert space. On the other hand, the NCSs can be regarded as particular cases of more general CSs, namely the so called generalized hypergeometric coherent states (GH-CSs) whose appellation becomes from their normalization function which is given by a generalized hypergeometric function [3], [4]. Let us denote by $z = |z| \exp\left(\mathrm{i}\,\phi\right)$, with $|z| \in \left(0, +\infty\right)$, and $\phi \in \left[0, 2\pi\right]$, the continuous parameter which labels the CSs and run over a complex domain. Moreover, any set of CSs must fulfill some conditions summarized by Klauder (called "the Klauder's prescriptions"): continuity in the complex label z, non-orthogonality, but normalization, unity operator resolution with positive defined integration measure, temporal stability and action identity [5].

In different calculations involving the CSs it is necessary to use some rules for the operator ordering. A useful and practical technique applicable to the canonical CSs related with the HO-1D, namely, the integration within an ordered product (IWOP) technique, was elaborated by H. −Y. Fan (see, e.g. [6] and references therein). Generally, for any pair of lowering L_- and raising L_+ operators, which generate the GH-CSs, previously we introduced a new operational technique, called the diagonal ordering operation technique (DOOT) and denoted with the symbols # # [7]. The main rules of DOOT are:

1. Inside the symbol # # the order of the operators L_- and L_+ can be permuted like commutable operators, but so that finally will result an operator function that depends only on the powers of normally ordered operator product $L_+ L_-$, i.e.

$$\#(L_-)^n(L_+)^n\# = \#\,(L_+)^n(L_-)^n\,\# = (L_+L_-)^n \tag{1.1}$$

2. A symbol $\#\ \ \#$ inside another symbol $\#\ \ \#$ can be deleted.
3. If the integration is convergent, a normally ordered product of operators can be integrated or differentiated, with respect to c-numbers, according to the usual rules. In addition, the c-numbers can be taken out from the symbol $\#\ \ \#$.
4. The projector $|\,0;\lambda><0;\lambda\,|$ of the normalized vacuum state $|\,0;\lambda>$, in the frame of DOOT, has the following normal ordered form:

$$|\,0;\lambda><0;\lambda\,| = \#\frac{1}{{}_pF_q\left(\left\{a_i\left(\lambda\right)\right\}_1^p;\left\{b_j\left(\lambda\right)\right\}_1^q;L_+L_-\right)}\# \tag{1.2}$$

Here ${}_pF_q\left(\left\{a_i\left(\lambda\right)\right\}_1^p;\left\{b_j\left(\lambda\right)\right\}_1^q;L_+L_-\right)$ is generalized hypergeometric function of integer orders p and q depending on the operator product "variable" L_+L_-. In order to shorten formulas, we have noted the real number sequence as $a_1,a_2,...,a_p \equiv \left\{a_i\left(\lambda\right)\right\}_1^p$ and so on, where λ is a parameter which characterizes CSs.

In the paper we apply the DOOT to the systems with non-equidistant levels, with the objective of broadening the area of applicability of this operator technique regarding the CSs. The main advantage of the DOOT is that it is not necessary to find the quantum group that governs the system into consideration, respectively to know the group generators.

2 Systems with Non-Equidistant Energy Levels

Let us consider a Hamiltonian H whose dimensionless energy eigenvalues e_n are expressed as a quadratic function with respect to the principal (main) quantum number n:

$$e_n = n\left(n+b\right) \tag{2.1}$$

where b is a real constant. If it is positive, the systems have an infinite number of eigenstates, while if b is negative, the number of eigenstates (respectively, the bound states) is finite, equal to the entire part of this number: $n_{\max} = \left[b/2\right]$. Evidently, in both situations, the energy spectra contain the energy levels which are non-equidistant.

Let us choose a pair of lowering L_- and raising L_+ hermitical operators acting on the Fock space vectors as

$$L_-\,|\,n;b> = \sqrt{n\left(n+b\right)}\,|\,n-1;b>,\ \ L_+\,|\,n;b> = \sqrt{\left(n+1\right)\left(n+1+b\right)}\,|\,n+1;b> \tag{2.2}$$

This choice is performed so that the normally ordered product of these operators is just the Hamiltonian of systems in consideration:

$$H\,|\,n;b> = L_+L_-\ \ |\,n;b> = n\left(n+b\right)|\,n;b> \equiv L\left(n\right)|\,n;b> \tag{2.3}$$

In the next, for shortening, we will use the following notation:

$$L\left(x\right) \equiv x\left(x+b\right) \tag{2.4}$$

Consequently, a Fock vector $|\,n;b>$ can be obtained by applying n- times the raising operator on the vacuum state $|\,0;b>$ and, similarly, the lowering operator on their dual $<n;b\,|$

$$|\,n;b> = \frac{1}{\sqrt{\left(1+b\right)_n n!}}\left(L_+\right)^n|\,0;b>,\ \ <n;b\,| = \frac{1}{\sqrt{\left(1+b\right)_n n!}}<0;b|\left(L_-\right)^n \tag{2.5}$$

where $\left(a\right)_n = \Gamma\left(n+a\right)/\Gamma\left(a\right)$ is the Pochhammer symbol, and $\Gamma\left(a\right)$ is the Euler Gamma function.

Beginning from the completeness relation for the Fock vectors basis $\left\{\,|\,n;b>,\ \ n=0,1,2,...\right\}$

$$\sum_{n=0}^{\infty}|\,n;b><n;b\,| = 1 \tag{2.6}$$

and using the DOOT rules, we can deduce the expression for the vacuum projector $|0;b><0;b|$:

$$
\begin{aligned}
\sum_{n=0}^{\infty}|n;b><n;b| &= \sum_{n=0}^{\infty}\frac{1}{\left(1+b\right)_n n!}\left(L_+\right)^n|0;b><0;b|\left(L_-\right)^n \# \\
&= \#|0;b><0;b|\# \#\sum_{n=0}^{\infty}\frac{1}{\left(1+b\right)_n n!}\left(L_+\right)^n\left(L_-\right)^n \# \\
&= \#|0;b><0;b|\# \sum_{n=0}^{\infty}\frac{1}{\left(1+b\right)_n}\frac{\#\left(L_+L_-\right)^n\#}{n!} \\
&= \#|0;b><0;b|\# \quad \#\,{}_0F_1(;1+b;L_+L_-)\,\# = 1
\end{aligned}
\tag{2.7}
$$

from which follows the expression of the projector of vacuum state (1.2), particularized for the examined case:

$$
|0;b><0;b| = \#\frac{1}{{}_0F_1(;1+b;L_+L_-)}\# \tag{2.8}
$$

3 Barut-Girardello Coherent States

As it is well-known, the Barut-Girardello coherent states (BG-CSs) are defined as being the eigenvalues of the lowering operator L_- [8]

$$
L_-|z;b> = z|z;b> \tag{3.1}
$$

The BG-CSs can be written in terms of the Fock – vector basis $\left\{|n;b>,\ n=0,1,2,...\right\}$ as

$$
|z;b> = \frac{1}{\sqrt{{}_0F_1(;1+b;|z|^2)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\left(1+b\right)_n n!}}|n;b> \tag{3.2}
$$

or, using the DOOT

$$
|z;b> = \frac{1}{\sqrt{{}_0F_1(;1+b;|z|^2)}}\,{}_0F_1(;1+b;zL_+)|0> \tag{3.3}
$$

The normalization confluent hypergeometric function ${}_0F_1(;1+b;|z|^2)$ was obtained by using the normalization condition for BG-CSs $<z;b|z;b>=1$.

The overlap (or scalar product) of two BG-CSs is

$$
<z;b|z';b> = \frac{{}_0F_1(;1+b;z^*z')}{\sqrt{{}_0F_1(;1+b;|z|^2)}\sqrt{{}_0F_1(;1+b;|z'|^2)}} \tag{3.4}
$$

from which we can see that the BG-CSs are *normalizable* but *non-orthogonal*.

Then, writing in a similar manner also the bra counterpart of BG-CSs $<z;b|$ and using Eq. (2.8) we can express the CSs projector

$$
|z;b><z;b| = \#\frac{1}{{}_0F_1(;1+b;L_+L_-)}\frac{{}_0F_1(;1+b;zL_+)\,{}_0F_1(;1+b;zL_-)}{{}_0F_1(;1+b;|z|^2)}\# \tag{3.5}
$$

where, if we put $z=0$, we recover the vacuum projector for BG-CSs.

The *continuity in the complex label*, in the sense that if $z'\to z$, then $|z';b>\to|z;b>$, is easy to be demonstrated:

$$
\lim_{z'\to z}\left\||z';b>-|z;b>\right\|^2 = 2-\lim_{z'\to z}\left(<z;b|z';b>+<z';b|z;b>\right) = 0 \tag{3.6}
$$

The resolution *of unity operator* (or the completeness relation), i.e.

$$
\int d\mu_{0,1}\left(z;b\right)|z;b><z;b| = 1 \tag{3.7}
$$

is accomplished if the integration measure $d\mu_{0,1}\left(z;b\right)$ which must be determined, is positive defined.

We try to find it by supposing the following structure:

$$d\mu_{0,1}\left(z;b\right)=\frac{d\phi}{2\pi}d\left(|z|^{2}\right)h_{0,1}\left(z;b\right) \tag{3.8}$$

Here we have inserted the indexes 0 and 1 in order to emphasize that these BG-CSs are in fact *one of particular cases* of the Barut-Girardello generalized hypergeometric coherent states (GH-BG-CSs) with the indexes $p=0$ and $q=1$, as well as the parameter $1+b$ [4].

We indicate only the main calculation steps. Substituting the projector (3.5), firstly we perform the function change for the weight function: $\tilde{h}_{0,1}\left(z;b\right)=h_{0,1}\left(z;b\right)\left[{}_0F_1(;1+b;|z|^2)\right]^{-1}$. Then we perform the angular integration:

$$\int_0^{2\pi}\frac{d\phi}{2\pi}\#\,{}_0F_1(;1+b;zL_+)\,{}_0F_1(;1+b;zL_-)\#=\sum_{n=0}^{\infty}\frac{\#\left(L_+L_-\right)^n\#}{\left[\left(1+b\right)_n n!\right]^2}\left(|z|^2\right)^n$$
$$={}_0F_3\left(;1,\,1+b,\,1+b\,;|z|^2\,L_+L_-\right) \tag{3.9}$$

Thus, it follows to solve a Stieltjes moment problem [9]

$$\int_0^{\infty}d\left(|z|^2\right)\tilde{h}_{0,1}\left(z;b\right)\left(|z|^2\right)^n=\left(1+b\right)_n n!=\frac{1}{\Gamma\left(1+b\right)}\Gamma\left(n+1+b\right)\Gamma\left(n+1\right) \tag{3.10}$$

Following a standard procedure, after an index change $n=s-1$ we obtain the result [9]

$$\tilde{h}_{0,1}\left(z;b\right)=\frac{1}{\Gamma\left(1+b\right)}G_{0,2}^{2,0}\left(|z|^2\middle|\,b,\quad 0\right)=\frac{2}{\Gamma\left(1+b\right)}|z|^b\,K_b\left(2\,|z|\right) \tag{3.11}$$

where $K_b\left(2\,|z|\right)$ is the modified Bessel function of the second kind.

This means that the integration measure finally is

$$d\mu_{0,1}\left(z;b\right)=\frac{2}{\Gamma\left(1+b\right)}\frac{d\phi}{2\pi}d\left(|z|^2\right)|z|^b\,K_b\left(2\,|z|\right){}_0F_1\left(;1+b;|z|^2\right) \tag{3.12}$$

It is evident that the weight function of the integration measure is a *positive defined function*. It can be verified without difficulty the correctness of this expression, using among other an integral involving the powers and the modified Bessel function of the second kind (see, e.g. [10], Eq. 6.564.16).

The expectation value of an operator A that characterizes the examined system is

$$<z;b|\,A\,|\,z;b>\equiv<A>_z=\frac{1}{{}_0F_1\left(;1+b;|z|^2\right)}\sum_{n,n'=0}^{\infty}\frac{\left(z^*\right)^n z^{n'}}{\sqrt{\left(1+b\right)_n n!\,\left(1+b\right)_{n'}n'!}}<n;b|A\,|\,n';b> \tag{3.13}$$

In practice, a privileged attention is paid to the diagonal operators in Fock basis. As an example, the expectation value of the normally ordered product of operators is

$$<z;b|\,L_+L_-\,|\,z;b>\equiv<L_+L_->_z=\frac{1}{{}_0F_1(;1+b;|z|^2)}\sum_{n=0}^{\infty}\frac{\left(|z|^2\right)^n}{\left(1+b\right)_n n!}L\left(n\right) \tag{3.14}$$

Using the property of Pochhammer symbols $\left(a\right)_n=\left(a+n-1\right)\left(a\right)_{n-1}$, the above sum becomes

$$\sum_{n=0}^{\infty}\frac{\left(|z|^2\right)^n}{\left(1+b\right)_n n!}L\left(n\right)=\sum_{n=0}^{\infty}\frac{\left(|z|^2\right)^n}{\left(1+b\right)_n n!}n\left(n+b\right)=|z|^2\sum_{n=0}^{\infty}\frac{\left(|z|^2\right)^{n-1}}{\left(1+b\right)_{n-1}(n-1)!} \tag{3.15}$$

Then, by using a new summation index $m=n-1$ and eliminating the unphysical term with $m=-1$, for the expectation value we obtain the result

$$<L_+L_->_z=|z|^2 \tag{3.16}$$

This result can be obtained in another, maybe more elegant manner. We transform the above sum as follows

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\left(|z|^2\right)^n}{\left(1+b\right)_n n!}L\left(n\right) &= |z|^2\frac{d}{d|z|^2}\left(|z|^2\frac{d}{d|z|^2}+b\right)\sum_{n=0}^{\infty}\frac{\left(|z|^2\right)^n}{\left(1+b\right)_n n!}\\ &= |z|^2\frac{d}{d|z|^2}\left(|z|^2\frac{d}{d|z|^2}+b\right){}_0F_1\left(;1+b;|z|^2\right)\\ &= L\left(|z|^2\frac{d}{d|z|^2}\right){}_0F_1\left(;1+b;|z|^2\right)\end{aligned} \tag{3.17}$$

An operator like $L\left(|z|^2\frac{d}{d|z|^2}\right)$ can be written as (where $|z|^2\equiv x$)

$$L\left(x\frac{d}{dx}\right)=x\frac{d}{dx}\left(x\frac{d}{dx}+b\right)=x\left[x\frac{d^2}{dx^2}+\left(b+1\right)\frac{d}{dx}\right] \tag{3.18}$$

On the other hand, the confluent hypergeometric function ${}_0F_1\left(;1+b;x\right)$ satisfies the following differential equation [11]:

$$\left[x\frac{d^2}{dx^2}+\left(b+1\right)\frac{d}{dx}-1\right]{}_0F_1\left(;1+b;x\right)=\left[\frac{1}{x}L\left(x\frac{d}{dx}\right)-1\right]{}_0F_1\left(;1+b;x\right)=0 \tag{3.19}$$

Consequently, when acting *only* on the hypergeometric function ${}_0F_1\left(;1+b;x\right)$, the next operator is canceled: $\frac{1}{x}L\left(x\frac{d}{dx}\right)-1=0$, i.e. the following equality is valid: $L\left(x\frac{d}{dx}\right)=x$ and, for an integer positive s, we have $\left[L\left(x\frac{d}{dx}\right)\right]^s=x^s$. Then, we obtain successively

$$\begin{aligned}&<\left(L_+L_-\right)^s>_z=\frac{1}{{}_0F_1(;1+b;|z|^2)}\left[L\left(|z|^2\frac{d}{d|z|^2}\right)\right]^s{}_0F_1\left(;1+b;|z|^2\right)\\ &=\frac{1}{{}_0F_1(;1+b;|z|^2)}\left(|z|^2\right)^s{}_0F_1\left(;1+b;|z|^2\right)\\ &=\left(|z|^2\right)^s\end{aligned} \tag{3.20}$$

For $s=1$ the above equation can be regarded also as an eigenvalue equation. In other words, $<L_+L_->_z=|z|^2$ is the eigenvalue of the operator $L\left(|z|^2\frac{d}{d|z|^2}\right)$ associated with the eigenfunction ${}_0F_1\left(;1+b;|z|^2\right)$.

Generally, for a function depending on the normally ordered operator product L_+L_- we have

$$<f\left(L_+L_-\right)>_z=\sum_{l=0}^{\infty}c_l<\left(L_+L_-\right)^l>_z=\sum_{l=0}^{\infty}c_l<\left(|z|^2\right)^l>_z=f\left(|z|^2\right) \tag{3.21}$$

This result means that if we have to calculate the expectation value, in the BG-CSs representation, of a function depending on the normally ordered operator product L_+L_-, the following *rule* is valid: it is sufficient to replace this operator product by their eigenvalue $|z|^2$.

An integer positive power s of a number operator $N|n;b>=n|n;b>$ can be expressed, using the DOOT, as follows:

$$\begin{aligned}N^s&=\sum_{n=0}^{\infty}n^s|n;b><n;b|=\#\frac{1}{{}_0F_1(;1+b;L_+L_-)}\sum_{n=0}^{\infty}\frac{\left(L_+L_-\right)^n}{\left(1+b\right)_n n!}n^s\#\\ &=\#\frac{1}{{}_0F_1(;1+b;L_+L_-)}\left(L_+L_-\frac{\partial}{\partial L_+L_-}\right)^s{}_0F_1\left(;1+b;L_+L_-\right)\#\end{aligned} \tag{3.22}$$

Then, according to the above rule, their expectation value is

$$< N^s >_z = \frac{1}{{}_0F_1\left(;1+b;|z|^2\right)}\left(|z|^2 \frac{\partial}{\partial |z|^2}\right)^s {}_0F_1\left(;1+b;|z|^2\right) \tag{3.23}$$

These expectations are useful to calculate the Mandel parameter $Q_{|z|^2}$ which is a powerful instrument to determine the statistical behavior of the BG-CSs [12]:

$$Q_{|z|} = \frac{< N^2 >_{z;S} - \left(< N >_{z;S}\right)^2}{< N >_{z;S}} - 1 = |z|^2 \left[\frac{{}_0F_1^{(2)}(;1+b;|z|^2)}{{}_0F_1^{(1)}(;1+b;|z|^2)} - \frac{{}_0F_1^{(1)}(;1+b;|z|^2)}{{}_0F_1(;1+b;|z|^2)}\right] \tag{3.24}$$

where we have used the notation

$${}_0F_1^{(s)}(;1+b;|z|^2) \equiv \left(\frac{d}{d|z|^2}\right)^s {}_0F_1(;1+b;|z|^2) = \frac{\Gamma(1+b)}{\Gamma(1+b+s)}\, {}_0F_1(;1+b+s;|z|^2) \tag{3.25}$$

The behavior of the above defined BG-CSs, can be sub-Poissonian (if $Q_{|z|^2} < 0$), Poissonian (if $Q_{|z|^2} = 0$) or super- Poissonian (if $Q_{|z|^2} > 0$), depending of the Mandel parameter $Q_{|z|^2}$ values, with respect to the variable $|z|^2$, respectively on the analytical properties of the expressions involving function ${}_0F_1(;1+b;|z|^2)$ and their derivatives. In order to evaluate this behavior we use the representation of the modified Bessel function of the first kind $I_a(2\sqrt{x}) = \frac{1}{\Gamma(a+1)}\left(\sqrt{x}\right)^a {}_0F_1(;1+a;x)$ [13]. Then, it follows

$$\begin{aligned} {}_0F_1^{(s)}(;1+b;|z|^2) &\equiv \left(\frac{d}{d|z|^2}\right)^s {}_0F_1(;1+b;|z|^2) \\ &= \frac{1}{\left(1+b\right)_s}\, {}_0F_1(;1+b+s;|z|^2) = \Gamma(1+b)\frac{I_{b+s}(2|z|)}{\left(|z|\right)^{b+s}} \end{aligned} \tag{3.26}$$

This leads to the following expression of Mandel parameter

$$Q_{|z|} = |z| \left[\frac{I_{b+2}(2|z|)}{I_{b+1}(2|z|)} - \frac{I_{b+1}(2|z|)}{I_b(2|z|)}\right] \le 0 \tag{3.27}$$

thanks to the inequality $I_{b+2}(2|z|)\, I_b(2|z|) \le \left[I_{b+1}(2|z|)\right]^2$ [14]. This means that these BG-CSs have a sub-Poissonian behavior.

The probability to occupy the n-th Fock state in the BG-CSs $|z;b>$ or the weighting distribution corresponding to BG-CSs is

$$\mathrm{P}_n\left(z;b\right) = |< n;b\,|\,z;b >|^2 = \frac{1}{{}_0F_1(;1+b;|z|^2)} \frac{\left(|z|^2\right)^n}{\frac{\Gamma(1+b+n)}{\Gamma(1+b)} n!} \tag{3.28}$$

For the *unphysical* limit $b \to \infty$ this distribution becomes the standard Poisson distribution $\mathrm{P}_n^{(P)}\left(z;b\right)$ with the shape parameter $|z|^2 \left(1+b\right)^{-1}$:

$$\lim_{b\to\infty} \mathrm{P}_n\left(z;b\right) = \mathrm{P}_n^{(P)}\left(z;b\right) = \exp\left(-\frac{1}{1+b}|z|^2\right) \frac{\left(\frac{1}{1+b}|z|^2\right)^n}{n!} \tag{3.29}$$

In order to calculate this relation we have used the limit [10] (Eq. 8.328.2): $\lim_{x\to\infty} \frac{\Gamma(x+a)}{\Gamma(x)} x^{-a} = 1$.

Let us apply the DOOT to the canonical density operator ρ which characterizes the mixed states of a quantum system in thermodynamical equilibrium with the environment at temperature $T = \left(\beta\, k_B\right)^{-1}$ where k_B is the Boltzmann constant. Their expression is

$$\begin{aligned}\rho &= \frac{1}{Z\left(\tilde{\beta}\right)}\exp\left(-\tilde{\beta}\,H\right) = \frac{1}{Z\left(\tilde{\beta}\right)}\#\exp\left(-\tilde{\beta}\,L_+L_-\right)\# \\ &= \frac{1}{Z\left(\beta\right)}\sum_{n=0}^{\infty}\exp\left[-\tilde{\beta}\;n\left(n+b\right)\right]\mid n;b><n;b\mid\end{aligned} \tag{3.30}$$

where $Z\left(\tilde{\beta}\right)$ is the partition function which will be determined in the next.

We transform the exponential according to the following *ansatz* (we have used the dimensionless quantity $\tilde{\beta} \equiv \beta\,\hbar\,\omega$):

$$\begin{aligned}\exp\left[-\tilde{\beta}\,n\left(n+b\right)\right] &= \left(e^{-\tilde{\beta}\,b}\right)^n\sum_{j=0}^{\infty}\frac{\left(-\tilde{\beta}\right)^j}{j!}n^{2j} = \sum_{j=0}^{\infty}\frac{\left(-\tilde{\beta}\right)^j}{j!}n^{2j}\left(e^{-\tilde{\beta}\,b}\right)^n \\ &= \sum_{j=0}^{\infty}\frac{\left(-\tilde{\beta}\right)^j}{j!}\frac{1}{b^{2j}}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^{2j}\left(e^{-\tilde{\beta}\,b}\right)^n = \exp\left[-\frac{1}{b^2}\tilde{\beta}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]\left(e^{-\tilde{\beta}\,b}\right)^n\end{aligned} \tag{3.31}$$

Previous we have introduced this *ansatz* in order to examine some properties of the Gazeau – Klauder quasi – coherent states for the Morse oscillator [15].

Then, the normalized canonical density operator becomes

$$\rho = \frac{1}{Z\left(\tilde{\beta}\right)}\exp\left[-\frac{1}{b^2}\tilde{\beta}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]\sum_{n=0}^{\infty}\left(e^{-\tilde{\beta}\,b}\right)^n\mid n;b><n;b\mid \tag{3.32}$$

Now, if we use the operator properties through the DOOT, we obtain

$$\rho = \frac{1}{Z\left(\tilde{\beta}\right)}\#\frac{1}{{}_0F_1(;\,1+b;L_+L_-)}\;\exp\left[-\frac{1}{b^2}\tilde{\beta}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]{}_0F_1(;\,1+b;L_+L_-\,e^{-\tilde{\beta}\,b})\# \tag{3.33}$$

Comparing this expression with the left one in Eq. (3.30) and using Eqs. (2.5) and (2.8), we obtain an interesting operator identity

$$\#\exp\left(-\tilde{\beta}L_+L_-\right){}_0F_1(;\,1+b;L_+L_-)\# = \#\exp\left[-\frac{1}{b^2}\tilde{\beta}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]{}_0F_1(;\,1+b;L_+L_-\,e^{-\tilde{\beta}\,b})\# \tag{3.34}$$

Using the DOOT and the above rules regarding the expectation values, the Q-distribution function (or, the Husimi's function), which is defined as the diagonal elements of the normalized density operator in the CSs representation [16], particularly for the BG-CSs is

$$\begin{aligned}Q\left(\mid z\mid^2;b\right) &\equiv <z;b\mid\rho\mid z;b> \\ &= \frac{1}{Z\left(\tilde{\beta}\right)}\frac{1}{{}_0F_1(;\,1+b;\mid z\mid^2)}\;\exp\left[-\frac{1}{b^2}\tilde{\beta}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]{}_0F_1(;\,1+b;\mid z\mid^2\,e^{-\tilde{\beta}\,b})\end{aligned} \tag{3.35}$$

It is not difficult to prove that the Q-distribution function is normalized to unity:

$$\int d\mu_{0,1}(z;b)\,Q\left(\mid z\mid^2;b\right) = 1 \tag{3.36}$$

During this calculation we get at the following relationship

$$1 = \frac{1}{Z\left(\tilde{\beta}\right)}\exp\left[-\frac{1}{b^2}\tilde{\beta}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]\sum_{n=0}^{\infty}\left(e^{-\tilde{\beta}\,b}\right)^n \tag{3.37}$$

and using Eq. (3.31) we get to the correct expression of the partition function

$$Z\left(\tilde{\beta}\right) = \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\tilde{\beta}\, n(n+b)}$$
$$= \exp\left[-\frac{1}{b^2}\tilde{\beta}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]\sum_{n=0}^{\infty}\left(e^{-\tilde{\beta}\, b}\right)^n = \exp\left[-\frac{1}{b^2}\tilde{\beta}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]\frac{1}{1-e^{-\tilde{\beta}\, b}} \tag{3.38}$$

The normalized canonical density operator can be expanded in terms of the BG-CSs projectors as

$$\rho = \frac{1}{Z\left(\tilde{\beta}\right)}\int d\mu_{0,1}(z;b)\, P_{0,1}(|z|^2;b)\,|z;b><z;b| \tag{3.39}$$

In order to determine the quasi-distribution function $P_{0,1}(|z|^2;b)$ we use Eq. (3.33) in the left hand side and the projector (3.5) in the right hand side. After performing the angular integration like (3.9), we will search the quasi-distribution function $P_{0,1}(|z|^2;b)$ having the following structure:

$$P_{0,1}(|z|^2;b) = \exp\left[-\frac{1}{b^2}\tilde{\beta}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right] R_{0,1}(|z|^2;b) \tag{3.40}$$

Using a standard procedure as for Eq. (3.10), yet we have to solve the Stietjels moment problem:

$$\int_0^{\infty} d\left(|z|^2\right)\tilde{R}_{0,1}\left(z;b\right)\left(|z|^2\right)^{n+\frac{b+1}{2}} = \left(1+b\right)_n\, n! = \frac{1}{2}\left(e^{-\tilde{\beta}\, b}\right)^{\frac{b+3}{2}}\frac{1}{\left(e^{-\tilde{\beta}\, b}\right)^s}\Gamma\left(s+\frac{b-1}{2}\right)\Gamma\left(s-\frac{b+1}{2}\right) \tag{3.41}$$

After solving this equation, the final result is

$$P_{0,1}(|z|^2;b) = \frac{1}{K_b\left(2|z|\right)}\exp\left[-\frac{1}{b^2}\tilde{\beta}\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]\left[\left(\sqrt{e^{\tilde{\beta}\, b}}\right)^{b+2} K_b\left(2|z|\sqrt{e^{\tilde{\beta}\, b}}\right)\right] \tag{3.42}$$

The quasi-distribution function $P_{0,1}(|z|^2;b)$ is also normalized to unity

$$\frac{1}{Z\left(\tilde{\beta}\right)}\int d\mu_{0,1}(z;b)\, P_{0,1}\left(|z|^2;b\right) = 1 \tag{3.43}$$

4 Some Applications

Let us particularize the above results to some physical systems with non-equidistant energy levels, namely the systems which have a quadratic dependence with respect to the main quantum number n. We will only indicate the way ahead, without make calculations and comprehensive exemplifications.

Firstly, we examine two cases of systems for which the constant b defining the energy eigenvalues is strictly positive: *a.*) the system with Hamiltonian whose eigenfunctions are the generalized Laguerre functions [2], respectively *b.*) the Pöschl-Teller like potential examined (through the factorization method) in [1].

For the *case a.*), i.e. *the system with Hamiltonian whose eigenfunctions are the generalized Laguerre functions* the constant b is a positive number, just the superior index of the generalized Laguerre polynomial $L_n^b(x)$ and the above *obtained* results can be given as such.

For the *case b.*), i.e. *the Pöschl-Teller like potential* (PT-like) is [1]

$$V\left(x\right) = V_0 \tan^2\left(\frac{\pi x}{L}\right),\ x \in \left[-\frac{L}{2}, +\frac{L}{2}\right] \tag{4.1}$$

where V_0 and L are two positive constants. The energy eigenvalues of the time - independent Schrödinger equation for a particle of mass m moved in the above PT-like potential are

$$E_n = \frac{\pi^2\hbar^2}{2mL^2}\left(n+\lambda\right)^2 \equiv E_0 + \frac{\pi^2\hbar^2}{2mL^2} e_n \tag{4.2}$$

where $e_n = n\left(n+2\lambda\right)$ and the constant λ is the positive solution of the equation $\lambda\left(\lambda-1\right)=\dfrac{2\,m\,L^2}{\pi^2\hbar^2}V_0$, i.e. $\lambda=\dfrac{1}{2}\left(1+\sqrt{1+\dfrac{8\,m\,L^2}{\pi^2\hbar^2}V_0}\right)$.

Consequently, for the PT-like potential we must take $b=2\lambda$, but, where appropriate, we should also take into account the zero energy $E_0=\dfrac{\pi^2\hbar^2}{2\,m\,L^2}\lambda^2$.

The next examined *case c.*) of *potentials with quadratic energy eigenvalues is the continuously indexed family of Pöschl-Teller potentials* with two characteristic constants $\lambda, k \geq 1$, and a period length $a>0$ [17]:

$$V\left(x\right)=\begin{cases} \dfrac{\hbar^2}{8\,m\,a^2}\left[\dfrac{\lambda\left(\lambda-1\right)}{\cos^2\left(\dfrac{x}{2\,a}\right)}-\dfrac{k\left(k-1\right)}{\sin^2\left(\dfrac{x}{2\,a}\right)}\right], & x\in\geq\left[0,\ \pi\,a\ \right] \\ \infty & x\leq 0,\quad x\geq \pi\,a \end{cases} \tag{4.3}$$

The dimensionless eigenenergies of this family of potentials are $e_n = n\left(n+\lambda+k\right)$ [18]. Consequently, the constant b , of our approach is $b=\lambda+k$ and all obtained results can be easy considered for this case.

If in above potentials family we take $\lambda=k=0$, we obtain the *case d.*), i.e. the *infinite rectangular well* who's dimensionless energy eigenvalues are $e_n = n^2$ [1]. This means that in the above formulae we must put $b=0$. In this case, the normalization function is [13] ${}_0F_1(;1;|\,z\,|^2)=I_0(2\,|\,z\,|)$. Consequently, the BG-CSs for the infinite rectangular well can be written as [19]:

$$|\,z;0>=\frac{1}{\sqrt{I_0\left(2\,|\,z\,|\right)}}\sum_{n=0}^{\infty}\frac{z^n}{n!}\,|\,n;0>=\frac{1}{\sqrt{I_0\left(2\,|\,z\,|\right)}}I_0\left(2\sqrt{zL_+}\right)|\,0;0> \tag{4.4}$$

where $|\,0;0>$ is the vacuum state.

The BG-CSs propagator is, then

$$|\,z;0><z;0\,|=\#\frac{1}{I_0\left(2\sqrt{L_+L_-}\right)}\frac{I_0\left(2\sqrt{z\,L_{+-}}\right)I_0\left(2\sqrt{z^*L_-}\right)}{I_0\left(2\,|\,z\,|\right)}\# \tag{4.5}$$

from which result also the vacuum projector.

The integration measure which result if we customize the constant $b=0$,

$$d\mu_{0,1}\left(z;0\right)=2\frac{d\phi}{2\pi}d\left(|\,z\,|^2\right)\,K_0\left(2\,|\,z\,|\right)\,I_0\left(2\,|\,z\,|\right) \tag{4.6}$$

ensure the completeness relation or the unity operator decomposition.

The expectation value satisfy the earlier deduced equation $<f\left(L_+L_-\right)>_z=f\left(|\,z\,|^2\right)$, and the Mandel parameter $Q_{|z|}$ is negative, so the BG-CSs for the infinite rectangular well is subject to a sub-Poissonian statistics.

5 Concluding Remarks

In the paper we have examined the quantum systems which have non-equidistant (quadratic) energy levels, in the frame of Barut-Girardello coherent states (BG-CSs), by using an earlier new introduced technique of ordering operators − *the diagonal ordering operation technique* (DOOT). The main advantage of this earlier proposed operator calculation technique (DOOT) is that it allows a unitary examination of the BG-CSs connected with *all* systems with non-equidistant quadratic energy levels *without the necessity to know explicitly the group generators of the quantum group associated with* *examined system.* It is sufficient to know the dimensionless energy eigenvalues e_n which are expressed as a function depending on the principal quantum number n.

In the frame of BG-CSs approach, the normally ordered product operator $\# L_+ L_- \#$ can be regarded as an operator "variable" corresponding to the numerical variable $|z|^2$. This manner facilitates algebraic calculations (especially integrations) on coherent states in the sense that, if we have to calculate the expectation values in the BG-CSs representation, it is sufficient to replace the normally ordered operator product $\# L_+ L_- \#$ with the numerical variable $|z|^2$.

The use of the DOOT allows obtaining new connections concerning hypergeometric functions. As an example, using Eq. (3.9), we obtain

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \# {}_0F_1(;\, a; z\, A)\, {}_0F_1(;\, 1+b; z\, B) \# = \sum_{n=0}^{\infty} \frac{\left(A\, B\right)^n}{\left[\left(a\right)_n n!\right]^2} \left(|z|^2\right)^n = {}_0F_3\left(;1,\, a,\, a\,; |z|^2\, A\, B\right) \tag{5.1}$$

On the other hand, through the present paper we have enlarged the applicability area of the previous introduced diagonal ordering operation technique (DOOT). Namely, the DOOT can be applied for a lot of quantum systems, not only to those having an infinite number of bound states (harmonic oscillator, pseudoharmonic oscillator [7], infinite quantum well [19]), but also for the systems with finite number of bound states, e.g. Morse oscillator [20], or spin systems [21] (in this last paper we showed that, by using the DOOT, the spin coherent states can also be constructed in the Barut-Girardello manner, not only in the Klauder-Perelomov manner, as traditionally (see, [22], [23] and references therein). All coherent states for these systems can be considered as the particular cases of more general coherent state, namely the generalized Barut-Girardello hypergeometric coherent states (GH-BG-CSs), whose names comes from the fact that their normalization functions are just the generalized hypergeometric functions.

The DOOT can be applied also in order to construct the CSs in the Gazeau-Klauder manner [24]. Moreover, by the help of DOOT, we have showed that the coherent states for the continuous spectrum can be regarded as the limiting case of hypergeometric Barut-Girardello coherent states (GH-BG-CSs) [25].

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