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Self-gravitating Envelope Solitons in Astrophysical Objects

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Abstract The propagation of ion-acoustic waves (IAWs) in a collisionless unmagnetized self-gravitating degenerate quantum plasma system (SG-DQPS) has been studied theoretically for the first time. A nonlinear Schrödinger equation is derived by using the reductive perturbation method to study the nonlinear dynamics of the IAWs in the SG-DQPS. It is found that for $k_c > k$ ($k_c < k$) (where k_c is critical value of the propagation constant k which determines the stable and unstable region of IAWs) the IAWs are modulationally unstable (stable), and that k_c depends on the ratio of the electron number density to light ion number density. It is also observed that the self-gravitating bright envelope solitons are modulationally stable. The results obtained from our present investigation are useful for understanding the nonlinear propagation of the IAWs in astrophysical compact objects like white dwarfs and neutron stars.

Keywords: Modulational instability, reductive perturbation method, envelope solitons.

1 Introduction

Recently, the self-gravity of degenerate quantum plasma (DQP) is the cornerstone among the plasma physicists to understand the basic features of the astrophysical compact objects (viz. white dwarf and neutron stars[1-7]). The self-gravitating DQP system (SG-DQPS) has a large number of ultra-relativistic or non-relativistic degenerate species (order of $10^{30}cm^{-3}$ in white dwarfs and order of $10^{36}cm^{-3}$ even more in neutron stars [3-6]) and extremely low temperature which exhibits unique collective behaviours from others plasma system. The basic constituents of the SG-DQPS (viz. white dwarf and neutron stars) are degenerate inertialess electron species [1-4], degenerate inertial light ion species (viz. $^1_1$H [8,9], $^4_2$He [1,2], and $^{12}_6$C [4,6]), and heavy ion species (viz. $^{56}_{26}$Fe [10], $^{85}_{37}$Rb [11], and $^{96}_{42}$Mo [11].).

The dynamics of the SG-DQPS is governed by the quantum mechanics because of the de Broglie wavelength of particles is comparable to the inter-particle distance in SG-DQPS [3,4]. According to the Heisenberg's uncertainty principle, in quantum realm, the exact position and momentum of a particle cannot be determined simultaneously, and mathematically it can be expressed as $\Delta x\Delta p \geq \hbar/2$ (where Δx is the uncertainty in position of the particle and Δp is the uncertainty in momentum of the same particle, and $\hbar$ is the reduced Planck constant). In SG-DQPS, the position (momentum) of the plasma species is well (not well) defined and these confined plasma species with uncertain momentum exerts a pressure on the surrounding medium. Chandrasekhar more than 80 years ago defined this exert pressure as degenerate pressure and mathematically it can be expressed as [1,2]

$$P_j = K_j N_j^{\gamma}, \qquad K_j \simeq \frac{3}{5}\frac{\pi\hbar^2}{m_j}, \tag{1}$$

where $j = e$ for the electron species, $j = l$ (h) for light (heavy) ion species, K_j is the proportional constant, γ is a relativistic factor and $\gamma = 5/4$ (5/3) stands for ultra-relativistic (non-relativistic) limit, and m_j is the mass of the plasma species. The degenerate pressure of the SG-DQPS is dependent (independent) on the number density and mass (temperature) of the plasma species. The mass of the plasma species generates a strong gravitational field which provides the inward pull to compress the plasma system, but this inward pull is counter-balanced by the outward degenerate pressure.

The amplitude of the ion-acoustic waves (IAWs) is appeared to modulation due to wave-particle interaction, the nonlinear self-interaction of the carrier wave modes, interaction between low and high frequency modes [12,13]. The modulational instability (MI) and generation of the envelope solitons in any nonlinear and dispersive medium are governed by the the nonlinear Schrödinger (NLS) equation. Recently, a large number of authors have studied the nonlinear wave propagation in SG-DQPS. Asaduzzaman *et al.* [14] have investigated the nonlinear propagation of self-gravitational perturbation mode in a super dense DQP medium. Mamun [15] analyzed shock structures in a self-gravitating, multi-component DQP and found that the height and thickness of the shock structures are totally dependent on the dissipative and nonlinear coefficients. Chowdhury *et al.* [16] have reported that the MI of nucleus-acoustic waves (NAWs) in a DQP system and found that the MI growth rate of the unstable NAWs is significantly modified by the number density of nucleus species. Islam *et al.* [17] have studied envelope solitons in three component DQP. However to the best of our knowledge, no attempt has been made to study MI of the IAWs in SG-DQPS. Therefore, in the present work, we will derive a NLS equation by employing reductive perturbation method (RPM) to study the MI and formation of the envelope solitons in a SG-DQPS (containing inertialess degenerate electron species, inertial degenerate light as well as heavy ion species).

The rest of the manuscript is organized as follows: The basic governing equations of our plasma model is presented in Sec. 2. The derivation of NLS equation by using RPM is presented in Sec. 3. The stability of the IAWs and envelope solitons are examined in Sec. 4. A brief discussion is provided in Sec. 5.

2 Governing Equations

We consider a SG-DQPS comprising of inertialess degenerate electron species e, inertial degenerate light ion species l, and heavy ion species h, respectively. The detail information about light and heavy nuclei is presented in Table 1. The nonlinear dynamics of such a SG-DQPS is governed by the following equations

$$\frac{\partial P_e}{\partial X} = -m_e N_e \frac{\partial \tilde{\psi}}{\partial X}, \tag{2}$$

$$\frac{\partial N_l}{\partial T} + \frac{\partial}{\partial X}(N_l U_l) = 0, \tag{3}$$

$$\frac{\partial U_l}{\partial T} + U_l \frac{\partial U_l}{\partial X} = -\frac{\partial \tilde{\psi}}{\partial X} - \frac{1}{m_l N_l}\frac{\partial P_l}{\partial X}, \tag{4}$$

$$\frac{\partial^2 \tilde{\psi}}{\partial X^2} = 4\pi G\,[m_e N_e + m_l N_l + m_h N_h], \tag{5}$$

where $T(X)$ is the time (space) variable; P_e (P_l) is the degenerate pressure associated with degenerate electrons (light ions); m_e, m_l, and m_h is the mass of electrons, light, and heavy ions, respectively; N_e, N_l, and N_h is, respectively, the number densities of the electrons, light, and heavy ions; U_l is the light ion fluid speed; $\tilde{\psi}$ is the self-gravitational potential; G is the universal gravitational constant. We consider the SG-DQPS in which the charge densities of positive and negative plasma particle species fluctuate in such a way that the wave electric field always remains constant. Now, the quasi-neutrality condition at equilibrium can be expressed as

$$N_e = Z_l N_l + Z_h N_h, \tag{6}$$

where Z_l (Z_h) is the charge state of a light (heavy) ion. For the purposes of simplicity, we have considered the continuity and momentum balance equation for the inertial light ion species l. Now, by introducing normalized variables, specifically, $x = X/L_q$, $t = T/\omega_{jl}$, $n_l = N_l/n_{l0}$, $u_l = U_l/C_q$, $\tilde{\psi} = C_q^2\psi$, [where $C_q = \sqrt{\pi}\hbar n_{e0}^{1/3}/m_l$, $\omega_{jl} = 4\pi G m_l n_{lo}$, $L_q = C_q/\omega_{jl}$; n_{l0} (n_{e0}) is the equilibrium number densities of light ion species (electrons); ψ is the dimensionless self-gravitational potential]. After normalization, Eqs. (2)$-$(5) appear in the following form

$$\frac{\partial \psi}{\partial x} = -\frac{3}{2}\alpha \frac{\partial n_e^{2/3}}{\partial x}, \tag{7}$$

$$\frac{\partial n_l}{\partial t} + \frac{\partial}{\partial x}(n_l u_l) = 0, \tag{8}$$

$$\frac{\partial u_l}{\partial t} + u_l \frac{\partial u_l}{\partial x} = -\frac{\partial \psi}{\partial x} - \beta \frac{\partial n_l^{2/3}}{\partial x}, \tag{9}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \gamma_e(n_e - 1) - \gamma_l(n_l - 1), \tag{10}$$

where $\alpha = m_l/m_e$, $\beta = (3/2)\mu^{-2/3}$, $\mu = n_{e0}/n_{l0}$, $\gamma_e = \mu(1/\alpha + \gamma/Z_l)$, $\gamma_l = \gamma - 1$; $\gamma = Z_l m_h/Z_h m_l$ (which is larger than 1 for any set of light and heavy ion species). In γ_e, $1/\alpha \ll \gamma/Z_l$ (where $1/\alpha$ varies from $\sim 10^{-4}$ to $\sim 10^{-3}$, and γ/Z_l varies from ~ 0.1 to ~ 2.0), so $1/\alpha$ is negligible compared to γ/Z_l, and can be written as $\gamma_e \simeq \mu\gamma/Z_l$. For inertialess degenerate electron species, the expression [from Eq. (7)] for the number density is

$$n_e = \left[1 - \frac{2\psi}{3\alpha^2}\right]^{\frac{3}{2}} = 1 - \frac{1}{\alpha^2}\psi + \frac{1}{6\alpha^4}\psi^2 + \frac{1}{54\alpha^6}\psi^3 + \cdots. \tag{11}$$

Now, by substituting Eq. (11) into Eq. (10), and expanding up to third order in ψ, we get

$$\frac{\partial^2\psi}{\partial x^2} - \gamma_l + \gamma_l n_l = \gamma_1\psi + \gamma_2\psi^2 + \gamma_3\psi^3 + \cdots, \tag{12}$$

where $\gamma_1 = -\gamma_e/\alpha^2$, $\gamma_2 = \gamma_e/6\alpha^4$, and $\gamma_3 = \gamma_e/54\alpha^6$. We note that the terms on the right hand side of Eq. (12) is the contribution of electron.

Table 1. The values of γ when ${}^{1}_{1}$H [8,9], ${}^{4}_{2}$He [1,2], and ${}^{12}_{6}$C [4,6] are considered as the light ion species, and ${}^{56}_{26}$Fe [10], ${}^{85}_{37}$Rb [11], and ${}^{96}_{42}$Mo [11] are considered as the heavy ion species.

Light ion	**Heavy ion**	γ
${}^{1}_{1}$H [8,9]	${}^{56}_{26}$Fe [10]	2.16
	${}^{85}_{37}$Rb [11]	2.30
	${}^{96}_{42}$Mo [11]	2.28
${}^{4}_{2}$He [1,2]	${}^{56}_{26}$Fe [10]	1.08
	${}^{85}_{37}$Rb [11]	1.15
	${}^{96}_{42}$Mo [11]	1.14
${}^{12}_{6}$C [4,6]	${}^{56}_{26}$Fe [10]	1.08
	${}^{85}_{37}$Rb [11]	1.15
	${}^{96}_{42}$Mo [11]	1.14

3 Derivation of the NLS Equation

In order to demonstrate the MI and the basic features of IAWs in a SG-DQPS, we employ the RPM [18,19] in which independent variables are stretched as

$$\left.\begin{array}{l}\xi = \epsilon(x - v_g t),\\ \tau = \epsilon^2 t,\end{array}\right\} \tag{13}$$

hence, we have

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \epsilon v_g \frac{\partial}{\partial \xi} + \epsilon^2 \frac{\partial}{\partial \tau}, \tag{14}$$

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \xi}, \tag{15}$$

where ϵ is a small parameter and v_g is the real variable interpreted as the group velocity. Furthermore, the dependent variables n_l, u_l, and ψ can be expanded in power series of ϵ as

$$n_l = 1 + \sum_{m=1}^{\infty} \epsilon^{(m)} \sum_{l'=-\infty}^{\infty} n_{ll'}^{(m)}(\xi, \tau) \exp[il'(kx - \omega t)], \tag{16}$$

$$u_l = \sum_{m=1}^{\infty} \epsilon^{(m)} \sum_{l'=-\infty}^{\infty} u_{ll'}^{(m)}(\xi, \tau) \exp[il'(kx - \omega t)], \tag{17}$$

$$\psi = \sum_{m=1}^{\infty} \epsilon^{(m)} \sum_{l'=-\infty}^{\infty} \psi_{l'}^{(m)}(\xi, \tau) \exp[il'(kx - \omega t)], \tag{18}$$

where ω (k) corresponds to the angular frequency (wave number) of the carrier waves, respectively. Now, by replacing the Eqs. (13)−(18) into Eqs. (8), (9), and (12), and collecting all terms of similar power of ϵ, the first order ($m = 1$ with $l' = 1$) reduced equations can be represented as

$$n_{l1}^{(1)} = \frac{k^2}{S}\psi_1^{(1)}, \tag{19}$$

$$u_{l1}^{(1)} = \frac{k\omega}{S}\psi_1^{(1)}, \tag{20}$$

where $S = \omega^2 - \beta_1 k^2$ and $\beta_1 = 2\beta/3$. The linear dispersion relation can be obtained from the first-order equations in the form

$$\omega^2 = \frac{\gamma_l k^2}{k^2 + \gamma_1} + \beta_1 k^2. \tag{21}$$

The dispersion properties of IAWs for different values of μ are depicted in Fig. 1 and it may deduce that (a) the value of ω exponentially decreases with the increase of k; (b) on the other hand, the value of ω increases (decreases) with n_{e0} (n_{l0}). Next, the second-order ($m = 2$ with $l' = 1$) reduced equations are given by

$$n_{l1}^{(2)} = \frac{k^2}{S}\psi_1^{(2)} + \frac{2i\omega k(v_g k - \omega)}{S^2}\frac{\partial \psi_1^{(1)}}{\partial \xi}, \tag{22}$$

$$u_{l1}^{(2)} = \frac{k\omega}{S}\psi_1^{(2)} + \frac{i(\beta_1 k^2 + \omega^2)(v_g k - \omega)}{S^2}\frac{\partial \psi_1^{(1)}}{\partial \xi}, \tag{23}$$

with the compatibility condition

$$v_g = \frac{\partial \omega}{\partial k} = \frac{\gamma_l \omega^2 - (\omega^2 - \beta_1 k^2)^2}{\gamma_l k \omega}. \tag{24}$$

The amplitude of the second-order harmonics is found to be proportional to $|\psi_1^{(1)}|^2$

$$\left.\begin{aligned} n_{l2}^{(2)} &= C_1|\psi_1^{(1)}|^2, & n_{l0}^{(2)} &= C_4|\psi_1^{(1)}|^2,\\ u_{l2}^{(2)} &= C_2|\psi_1^{(1)}|^2, & u_{l0}^{(2)} &= C_5|\psi_1^{(1)}|^2,\\ \psi_2^{(2)} &= C_3|\psi_1^{(1)}|^2, & \psi_0^{(2)} &= C_6|\psi_1^{(1)}|^2, \end{aligned}\right\} \tag{25}$$

where

$$C_1 = \frac{3\omega^2 k^4 + 2C_3 k^2 S^2}{2S^2}, \quad C_2 = \frac{C_1 \omega S^2 - \omega k^4}{kS^2}, \quad C_3 = \frac{3\gamma_l \omega^2 k^4 - 2\gamma_2 S^3}{2S^3(4k^2 + \gamma_1) - 2\gamma_l k^2 S^2},$$

$$C_4 = \frac{2\omega v_g k^3 - \beta_2 k^4 + k^2\omega^2 + C_6 S^2}{S^2(v_g^2 - \beta_1)}, \quad C_5 = \frac{C_4 v_g S^2 - 2\omega k^3}{S^2}, \quad \beta_2 = \frac{\beta}{9},$$

$$C_6 = \frac{\gamma_l(2\omega v_g k^3 - \beta_2 k^4 + k^2\omega^2) - 2\gamma_2 S^2(v_g^2 - \beta_1)}{\gamma_1 S^2(v_g^2 - \beta_1) - \gamma_l S^2}.$$

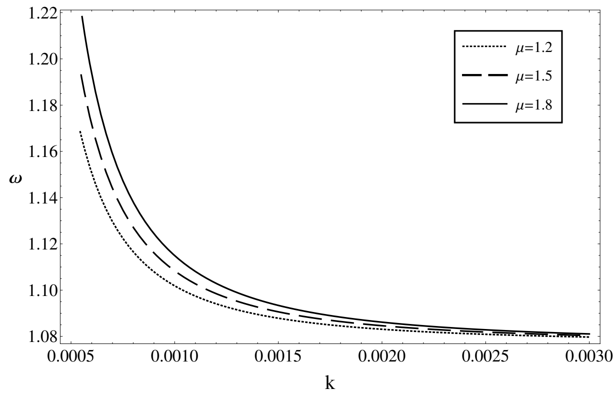

Figure 1. The variation of ω with k for different values of μ; along with $\alpha = 3.67 \times 10^3$, $\gamma = 2.16$, and $\gamma/Z_l = 0.5$.

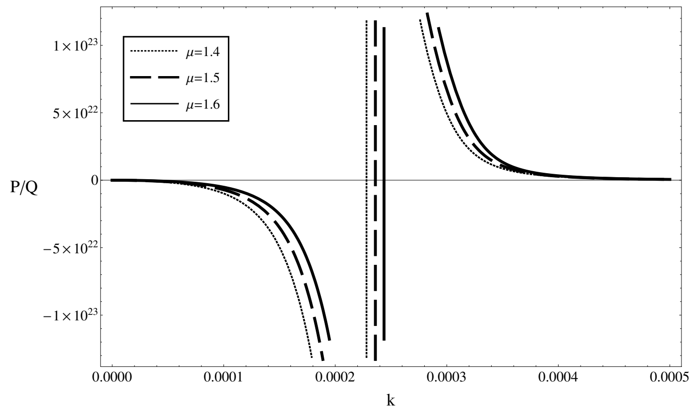

Figure 2. The variation of P/Q with k for different values of μ; along with $\alpha = 3.67 \times 10^3$, $\gamma = 2.16$, and $\gamma/Z_l = 0.5$.

Finally, the third harmonic modes ($m = 3$ with $l' = 1$) provide a set of equations and after some mathematical calculation these equations reduce [with the help of Eqs. (19)−(25)] to the following NLS equation:

$$i\frac{\partial \Phi}{\partial \tau} + P\frac{\partial^2 \Phi}{\partial \xi^2} + Q|\Phi|^2\Phi = 0, \tag{26}$$

where $\Phi = \psi_1^{(1)}$ for simplicity. The dispersion coefficient P is given by

$$P = \frac{v_g\beta_1^2k^5 + 4\beta_1k^2\omega^3 + 2\beta_1v_g\omega^2k^3 - 3v_gk\omega^4 - 4\omega\beta_1^2k^4}{2\gamma_lk^2\omega^2}, \tag{27}$$

and the nonlinear coefficient Q is given by

$$Q = \frac{1}{2\gamma_l\omega k^2S^2}\left[2\gamma_2(C_3 + C_6)S^4 + 2\gamma_3S^4 - F_1\right], \tag{28}$$

where $F_1 = \gamma_lk^2\omega^2S^2(C_1 + C_4) + 2\gamma_l\omega S^2k^3(C_2 + C_5) + \gamma_l\beta_3k^8$, and $\beta_3 = 4\beta/81$.

4 MI and Envelope Solitons

The MI of IAWs can be studied by considering the harmonic modulated amplitude solution of Eq. (26) of the form $\Phi = \hat{\Phi}e^{i\,Q|\hat{\Phi}|^2\tau} + c.\ c.$ (c. c. being the complex conjugate), where perturbed amplitudes are

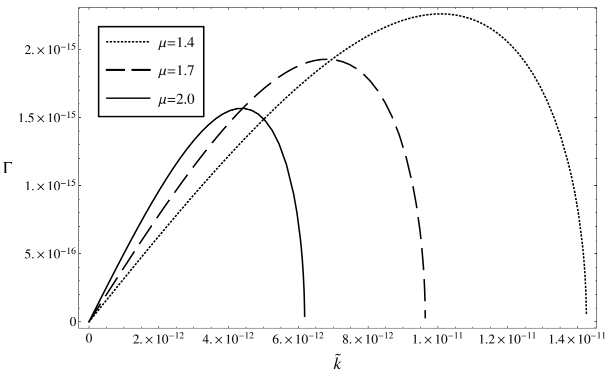

Figure 3. The variation of $\varGamma$ with $\tilde{k}$ for different values of μ; along with $\alpha = 3.67 \times 10^3$, $\gamma = 2.16$, $\gamma/Z_l = 0.5$, $k = 0.00035$, and $\varPhi_0 = 0.9$.

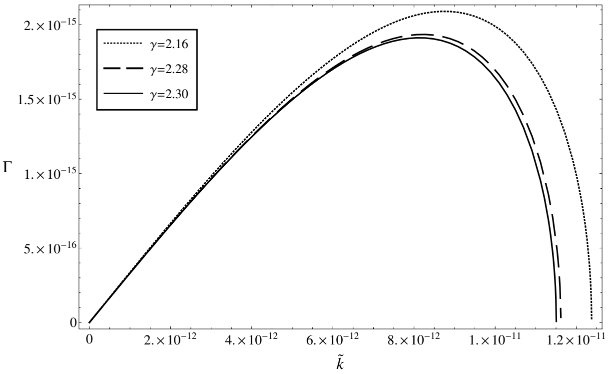

Figure 4. The variation of $\varGamma$ with $\tilde{k}$ for different values of γ; along with $\alpha = 3.67 \times 10^3$, $\gamma/Z_l = 0.5$, $\mu = 1.5$, $k = 0.00035$, and $\varPhi_0 = 0.9$.

$\hat{\varPhi} = \hat{\varPhi}_0 + \epsilon\hat{\varPhi}_1$ and $\hat{\varPhi}_1 = \hat{\varPhi}_{1,0} \exp[i(\tilde{k}\xi - \tilde{\omega}\tau)] + c.\ c$ (here, the perturbed wave number $\tilde{k}$ and the frequency $\tilde{\omega}$ are different from k and ω). Hence, the nonlinear dispersion relation for the amplitude modulation obtained by substituting these values in Eq. (26) can be written as [20-22]

$$\tilde{\omega}^2 = P^2 \tilde{k}^2 \left(\tilde{k}^2 - \frac{2|\hat{\varPhi}_o|^2}{P/Q} \right). \tag{29}$$

It is apparent from Eq. (29) that the IAWs will be modulationally stable (unstable) for the range of values of $\tilde{k}$ in which P/Q is negative (positive), such as, $P/Q < 0$ ($P/Q > 0$). When $P/Q \to \pm\infty$, the corresponding value of k ($= k_c$) is called threshold or critical wave number for the onset of MI. This k_c separates the unstable region ($P/Q > 0$) from the stable ($P/Q < 0$) one. The stability of the profile has been investigated by depicting the ratio of P/Q with carrier wave number k for different values of μ in Fig. 2, which clearly indicates that (a) there is an unstable (stable) region for IAWs for the large (small) value of k; (b) the k_c increases (decreases) with the increase of the value of n_{e0} (n_{lo}). So, the electron and ion number densities play an opposite role to recognize the stability domain of the IAWs. In the unstable region $P/Q > 0$ and under this certain condition $\tilde{k} < k_c = \sqrt{2Q|\hat{\varPhi}_o|^2/P}$, the growth rate ($\varGamma$) of MI is obtained from the Eq. (29) can be written as

$$\varGamma = |P|\ \tilde{k}^2 \sqrt{\frac{k_c^2}{\tilde{k}^2} - 1}, \tag{30}$$

where $\hat{\varPhi}_o$ is the amplitude of the carrier waves. We have numerically analysed the influence of different plasma parameters on the MI growth rate by depicting $\varGamma$ with $\tilde{k}$ [obtained from Eq.(30)] for different

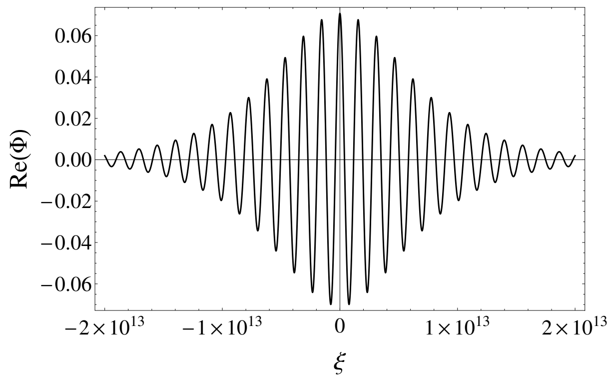

Figure 5. The variation of the $|\varPhi|$ with ξ for self-gravitating bright solitons; along with $\alpha = 3.67 \times 10^3$, $\gamma = 2.16$, $\mu = 1.5$, $k = 0.0003$, ψ_0 =0.005, $U = 0.001$, $\varOmega_0 = 0.04$, $\tau = 0$.

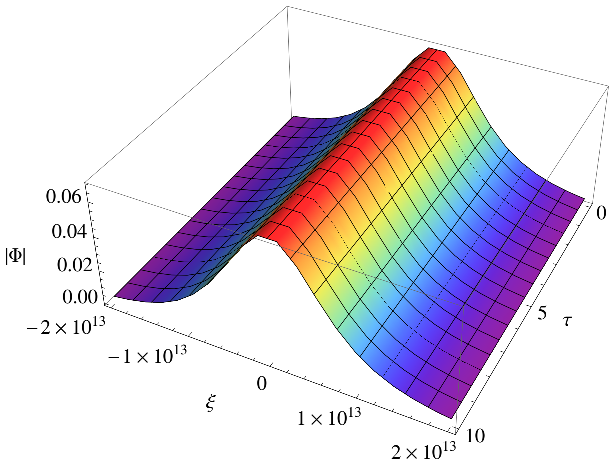

Figure 6. The variation of the $|\varPhi|$ with ξ and τ for self-gravitating bright solitons; along with $\alpha = 3.67 \times 10^3$, $\gamma = 2.16$, $\mu = 1.5$, $k = 0.0003$, ψ_0 =0.005, $U = 0.001$, $\varOmega_0 = 0.04$, $\tau = 0$.

values of μ and γ in Figs. 3 and 4 and it is obvious that (a) initially, the $\varGamma$ increases with $\tilde{k}$ before obtained it's maximum value $\varGamma_{max}$. But after $\varGamma_{max}$, the $\varGamma$ decreases to zero for further increase in $\tilde{k}$; (b) as we increase the value of the electron number density (n_{e0}), the maximum value of the growth rate decreases but increases with increases of the light ion number density (n_{l0}); (c) on the other hand, the growth rate $\varGamma$ decreases with the increase (decrease) of m_h (m_l) for fixed value of Z_l and Z_h (via $\gamma = Z_l m_h/Z_h m_l$); (d) similarly, $\varGamma$ decreases with the increase (decrease) of Z_l (Z_h) for fixed value of m_h and m_l (via γ). The physics of this result is that the maximum value of the growth rate increases as the nonlinearity of the plasma system increases with the increase (decrease) of the value of Z_h or m_l (Z_l or m_h).

The self-gravitating bright solitons are generated when the carrier wave is modulationally unstable in the region $P/Q > 0$, whose general analytical form reads as [20-22]

$$\varPhi(\xi,\tau) = \left[\psi_0 \ \mathrm{sech}^2\left(\frac{\xi - U\tau}{W}\right)\right]^{1/2} \times \exp\left[\frac{i}{2P}\left\{U\xi + \left(\varOmega_0 - \frac{U^2}{2}\right)\tau\right\}\right], \tag{31}$$

where U is the propagation speed, ψ_0 is the envelope amplitude, $\varOmega_0$ oscillating frequency for $U = 0$ and W is the soliton width which can be defined as $W = \sqrt{2|P/Q|/\psi_0}$. The self-gravitating bright envelope soliton depicted in Figs 5 and 6, clearly indicates that the shape of the self-gravitating bright envelope solitons is not affected by any external perturbation through the time evolution. So, the self-gravitating bright envelope solitons are modulationally stable.

5 Discussion

In this work, we have investigated the amplitude modulation of IAWs in an unmagnetized three components SG-DQPS comprising of inertialess degenerate electron species, inertial degenerate light and heavy ion species. A NLS equation is derived by employing the reductive perturbation method that governs the stability, circumstance for the appearance of MI growth rate, and formation of the IAWs envelope solitons in SG-DQPS. The noticeable results found from this theoretical investigation can be outlined as follows:

1. The ω increases (decreases) with n_{e0} (n_{l0}), and also decreases exponentially with the increase of k.
2. The IAWs will be stable (unstable) for smaller (larger) values of k.
3. The growth rate decreases with the increase of n_{e0} but it decreases with increase (decrease) of m_h (m_l) for fixed value of Z_l and Z_h, similarly, growth rate decreases with the increase (decrease) of Z_l (Z_h) for fixed value of m_h and m_l (via $\gamma = Z_l m_h/Z_h m_l$).
4. The shape of the self-gravitating bright envelope solitons is not affected by any external perturbation through the time evolution. So, the self-gravitating bright envelope solitons are modulationally stable.

In conclusion, we hope that the results from our present theoretical investigation may be helpful in understanding the nonlinear self-gravitating envelope solitons in astrophysical compact objects (viz. white dwarf and neutron stars [1-7]).

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