Theoretical Physics, Vol. 4, No. 1, March 2019
https://dx.doi.org/10.22606/tp.2019.41002

Energy in Special Relativity

Zbigniew Osiak

The University of the Third Age in the Wroclaw University, Wroclaw, Poland
Email: zbigniew.osiak@gmail.com

Abstract I give new relativistic formulas for kinetic, rest and total energies. The change in kinetic energy of a particle will be determined as the work done by the spatial part of the Minkowski four-force. The new relativistic kinetic energy and its relation with the spatial part of the four-momentum are given by the expressions similar to its classical equivalents. In particular, I justified that the rest energy is half of the amount determined by the traditional relation. I present a new interpretation of the temporal component of the Minkowski four-force.

Keywords: Special relativity.

1 Introduction

The traditional form of the mass-energy equivalence principle proposed by Einstein [1] $E_0 = mc^2$ is sometimes derived [2] using the Planck equations of motion [3] $\boldsymbol{F} = d(m\gamma\boldsymbol{v})/dt$. Usage of these equations in special relativity leads to errors because they are not covariant under Lorentz transformations. The Lorentz covariance of the physical equations and the Lorentz invariance of the speed of light in a vacuum are the two basic postulates of special relativity. In my view, the Planck equations of motion are an interesting example of a heuristic hypothesis and have only historical significance. For calculation of the relativistic kinetic energy and its relationship with the rest and the total energies it is necessary to use the Lorentz covariant four-dimensional Minkowski equations of motion [4].

2 The Lorentz Covariant Four-Dimensional Minkowski Equations of Motion

The Lorentz covariant four-dimensional Minkowski equations of motion are given by:

$$\tilde{F}_\alpha = m\gamma\frac{d\tilde{v}_\alpha}{dt} = m\tilde{a}_\alpha \tag{1}$$

where
$\tilde{F}_\alpha$ – components of the Minkowski four-force; $\alpha = 1, 2, 3, 4$; m – invariant mass of a particle; $\gamma \equiv \left(1 - v^2c^{-2}\right)^{-\frac{1}{2}}$ – Lorentz factor; c – speed of light in vacuum; $\tilde{v}_\alpha \equiv \gamma\left(dx_\alpha/dt\right)$ – components of the four-velocity; $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $x_4 \equiv ict$; i – imaginary unit; $\boldsymbol{v} \equiv d\boldsymbol{r}/dt$ – three-dimensional velocity; $\boldsymbol{r} \equiv (x, y, z) = (x_1, x_2, x_3)$ – three-dimensional position vector; $v^2 \equiv \left(dx_1/dt\right)^2 + \left(dx_2/dt\right)^2 + \left(dx_3/dt\right)^2$ – square modulus of the three-dimensional velocity; $\tilde{a}_\alpha \equiv \gamma\left(d\tilde{v}_\alpha/dt\right)$ – components of the four-acceleration.

3 The Spatial Part of the Minkowski Four-Force

The Minkowski four-force (1) can be written as

$$\tilde{\boldsymbol{F}} = \left(\boldsymbol{F}, m\gamma\frac{d\gamma ic}{dt}\right) \tag{2}$$

where

$$\boldsymbol{F} \equiv m\gamma\frac{d\gamma\boldsymbol{v}}{dt} \tag{3}$$

is the spatial part of this four-vector.

After taking into account that $d\gamma/dt = \gamma^3 c^{-2}\boldsymbol{v}\cdot(d\boldsymbol{v}/dt)$; $\boldsymbol{a} = d\boldsymbol{v}/dt$ – three-dimensional acceleration; $\boldsymbol{a} = \boldsymbol{a}_{\|} + \boldsymbol{a}_{\perp}$; $\boldsymbol{a}_{\|} \equiv \boldsymbol{a} \parallel \boldsymbol{v}$; $\boldsymbol{a}_{\perp} \equiv \boldsymbol{a}\perp\boldsymbol{v}$; $\boldsymbol{v}\cdot\boldsymbol{a}_{\perp} = 0$; $\boldsymbol{v}\cdot\left(\boldsymbol{v}\cdot\boldsymbol{a}_{\|}\right) = v^2\boldsymbol{a}_{\|}$; $v^2c^{-2} + \gamma^{-2} = 1$, we can write the spatial part of the Minkowski four-force (3) in the form needed for further considerations:

$$\boldsymbol{F} = m\gamma^4\boldsymbol{a}_{\|} + m\gamma^2\boldsymbol{a}_{\perp} \tag{4}$$

4 The New Formulas for Energies in Relativistic Mechanics

Let a particle of mass m moves (for simplicity) uniformly along a straight line with velocity $\boldsymbol{v}$. Now we will determine the kinetic energy of this particle, i.e. the work that needed to be done by the spatial part of the Minkowski four-force (3, 4) in order to accelerate initially resting particle to velocity $\boldsymbol{v}$ (in addition, we assumed that $\boldsymbol{v}$, $\boldsymbol{a}$ and $\boldsymbol{F}$ are parallel to $\boldsymbol{r}$).

$$E_k = \int_0^v \boldsymbol{F}\cdot d\boldsymbol{r} \tag{5}$$

Because $\boldsymbol{a}_{\|}\cdot d\boldsymbol{r} = \boldsymbol{a}\cdot d\boldsymbol{r}$; $\boldsymbol{a}_{\perp}\cdot d\boldsymbol{r} = 0$; $\boldsymbol{a} \equiv d\boldsymbol{v}/dt$; $d\boldsymbol{r} = \boldsymbol{v}dt$, the expression under the integral takes the form:

$$\boldsymbol{F}\cdot d\boldsymbol{r} = m\gamma^4\boldsymbol{v}\cdot d\boldsymbol{v} = m\gamma^4 v dv = dE_k \tag{6}$$

Finally, we can determine the relativistic kinetic energy of a particle:

$$E_k = \frac{1}{2}m\gamma^2c^2 - \frac{1}{2}mc^2 = \frac{1}{2}m\gamma^2v^2 \tag{7}$$

where

$$E \equiv \frac{1}{2}m\gamma^2c^2 \tag{8}$$

and

$$E_0 \equiv \frac{1}{2}mc^2 \tag{9}$$

are the total energy of a particle of mass m moving with velocity $\boldsymbol{v}$ and the rest energy of a particle, respectively.

The relativistic kinetic energy (7) of a particle moving at low speed in relation to the speed of light ($v \ll c$) is approximately equal to the value determined from the classical formula:

$$E_k \approx \frac{1}{2}mv^2 \tag{10}$$

Note that Einstein in his famous paper [5] proposed for the kinetic energy the following expression:

$$E_k = m\gamma c^2 - mc^2 \tag{11}$$

5 The Square Modulus of the Four-Velocity

The four-vector of velocity (four-velocity) is defined by:

$$\tilde{\boldsymbol{v}} \equiv (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) = \left(\gamma\frac{dx_1}{dt}, \gamma\frac{dx_2}{dt}, \gamma\frac{dx_3}{dt}, \gamma\frac{dx_4}{dt}\right) \tag{12}$$

Calculating the square modulus of this four-vector

$$\tilde{\boldsymbol{v}}^2 = \gamma^2v^2 - \gamma^2c^2 = -c^2 \tag{13}$$

we obtain the following equation:

$$\gamma^2v^2 + c^2 = \gamma^2c^2 \tag{14}$$

By multiplying both sides of the above equation by $\frac{1}{2}m$ we receive again the relationship between kinetic, rest and total energies of a particle moving with velocity $\boldsymbol{v}$:

$$\frac{1}{2}m\gamma^2v^2 + \frac{1}{2}mc^2 = \frac{1}{2}m\gamma^2c^2 \tag{15}$$

6 The New Relation between Kinetic Energy and Momentum

The four-vector of momentum (four-momentum) is defined through the equation:

$$\tilde{\boldsymbol{p}} \equiv m\tilde{\boldsymbol{v}} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4) \tag{16}$$

where

$$\tilde{p}_\alpha \equiv m\tilde{v}_\alpha \tag{17}$$

The temporal component of the four-momentum using Eqs. (17, 12, 8) takes the form:

$$\tilde{p}_4 \equiv m\tilde{v}_4 = m\gamma ic = i\sqrt{2mE} \tag{18}$$

Determining twice the square modulus of the four-momentum (16), we obtain:

$$\tilde{\boldsymbol{p}}^2 = -m^2c^2 \tag{19}$$

and

$$\tilde{\boldsymbol{p}}^2 = p^2 - m^2\gamma^2c^2 \tag{20}$$

where

$$\boldsymbol{p} \equiv m\gamma\boldsymbol{v} = \left(m\gamma\frac{dx_1}{dt}, m\gamma\frac{dx_2}{dt}, m\gamma\frac{dx_3}{dt}\right) = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \tag{21}$$

and

$$\boldsymbol{p}^2 = p^2 = \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 = m^2\gamma^2v^2 \tag{22}$$

are the spatial part of the four-momentum and the square of its modulus, respectively.

Equating the right sides of the both equations (19) and (20) for the square modulus of the four-momentum, after simple transformations and taking into account Eqs. (7, 8, 9), we have:

$$\frac{p^2}{2m} = \frac{1}{2}m\gamma^2c^2 - \frac{1}{2}mc^2 = E - E_0 \tag{23}$$

or

$$\frac{p^2}{2m} = \frac{1}{2}m\gamma^2v^2 = E_k \tag{24}$$

7 The Temporal Component of the Minkowski Four-Force

The temporal component of the Minkowski four-force

$$\tilde{F}_4 = m\gamma\frac{d\gamma ic}{dt} = imc\gamma\frac{d\gamma}{dt} \tag{25}$$

after taking into consideration that $\gamma d\gamma/dt = \left(d\gamma^2/dt\right)/2$ and Eqs. (7, 8, 9) may be written as:

$$\tilde{F}_4 = ic^{-1}\frac{d}{dt}\left(\frac{1}{2}m\gamma^2c^2\right) = ic^{-1}\frac{dE}{dt} = ic^{-1}\frac{dE_k}{dt} \tag{26}$$

Note that the dot product of the Minkowski force four-vector

$$\tilde{\boldsymbol{F}} = \left(\tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4\right) = \left(\boldsymbol{F}, \tilde{F}_4\right) \tag{27}$$

and the differential of the position four-vector

$$d\tilde{\boldsymbol{r}} = (dx_1, dx_2, dx_3, dx_4) = (d\boldsymbol{r}, dx_4) \tag{28}$$

is the Lorentz invariant equal to zero.

$$\tilde{\boldsymbol{F}} \cdot d\tilde{\boldsymbol{r}} = \tilde{F}_1dx_1 + \tilde{F}_2dx_2 + \tilde{F}_3dx_3 + \tilde{F}_4dx_4 = \boldsymbol{F} \cdot d\boldsymbol{r} + \tilde{F}_4dx_4 = 0 \tag{29}$$

8 A More General Form of the Mikowski Equations of Motion

Combining Eqs. (1) and (17), we obtain a more general form of the Minkowski equations of motion:

$$\tilde{F}_\alpha = \gamma \frac{dm\tilde{v}_\alpha}{dt} = \gamma \frac{d\tilde{p}_\alpha}{dt} \tag{30}$$

From Eq. (30) follows the conservation law of momentum and energy, which states that:
If all components of the four-force acting on the particle are equal to zero, then all components of the four-momentum of this particle are constant over time.

9 Conclusions and Discussion

Presented in this paper the new relativistic formulas for kinetic, rest and total energies differ from the commonly applied analogous expressions. These differences are caused by assumption that the Minkowski equations of motion are correct while the Planck equations of motion are incorrect. The new relativistic kinetic energy and its relation with the spatial part of the four-momentum are given by the expressions similar to its classical equivalents. In particular, I justified that the rest energy is half of the amount determined by the traditional relation $E_0 = mc^2$. The most popular formula of physics is not correct.

Disagreement between the traditional form of the mass-energy equivalence principle and the experimental data was recently discussed by Muhyedeen [6] .

References

1. A. Einstein, "Ist die Trägheit eines Körpers von seinem Energieinhalt abhänging?", *Annalen der Physik*, vol. 323, pp. 639–641, 1905.
2. R. Katz, "An Introduction to the Special Theory of Relativity", D. Van Nostrand Company, Inc., Princeton, NJ, 1964.
3. M. Planck, "Das Prinzip der Relativität und die Grundgleichungen der Mechanik", *Verhandlungen der Deutschen physikalischen Gesellschaft*, vol. 8, pp. 136–141, 1906. [See equation 6.]
4. H. Minkowski, "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern", *Nachrichten von der Königlich Gesellschaft der Wissenschaften zu Göttingen (Mathematisch-physikalische Klasse)*, pp. 53–111, 1908. [See equation 22 on page 107.]
5. A. Einstein, "Zur Elekrodynamik bewegter Körper", *Annalen der Physik*, vol. 32, pp. 891–921, 1905.
6. B. Muhyedeen, "New Concept of Mass-Energy Equivalence", *European Journal of Scientific Research*, vol. 26, pp. 161–175, 2009.